Technische Universität Berlin **Sommersemester 2012**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Systems and control theory

Series 4

Task 1:
Let $G \in \mathcal{C}_\infty^{n,m}$ and $t_0 < t_1$. Define the associated *Gramian* by

$$V(t_0, t_1) := \int_{t_0}^{t_1} G(s)G^*(s)ds \in \mathbb{C}^{n,n}.$$

1. Show that the Gramian is positive semi-definite, i.e., $\oplus$

$$V(t_0, t_1) = V^*(t_0, t_1) \geq 0.$$

2. Show that $\oplus$

$$\text{kernel}\,(V(t_0, t_1)) = \left\{ z_0 \in \mathbb{C}^n \;\middle|\; G^*(t) z_0 = 0 \text{ in } [t_0, t_1] \right\}.$$

3. Show that $\ominus$

$$\text{image}\,(V(t_0, t_1)) = \underbrace{\left\{ z_0 \in \mathbb{C}^n \;\middle|\; \exists u \in \mathcal{C}_\infty^m \text{ with } z_0 = \int_{t_0}^{t_1} G(s)u(s)ds \right\}}_{=:\mathfrak{V}}.$$

Remark: For 3.) first show that $\text{image}\,(V(t_0, t_1)) \subset \mathfrak{V}$. Then show that $\text{kernel}\,(V(t_0, t_1)) \cap \mathfrak{V} = \{0\}$. Finally, use dimension arguments from linear algebra to show that for every matrix $A \in \mathbb{C}^{n,n}$ and every linear space $\mathfrak{V} \subset \mathbb{C}^n$ which satisfy the relations $\text{image}\,(A) \subset \mathfrak{V}$ and $\text{kernel}\,(A) \cap \mathfrak{V} = \{0\}$ we have that $\mathfrak{V} = \text{image}\,(A)$.

Task 2: $\ominus$
Complete the proof of Theorem 2.4, i.e., show that the controllable subspace is the image of the Gramian of controllability.

Task 3: $\odot$
Let $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$. Show that the controllable and reachable sets of the time-**in**variant system $\dot{x} = Ax + Bu$ only depend on $t_1 - t_0$, i.e., show that

$$\begin{aligned} C(t_1, t_0) &= C(t_1 - t_0, 0) \\ R(t_1, t_0) &= R(t_1 - t_0, 0), \end{aligned}$$

for any $t_0 < t_1$ as claimed before Definition 2.5.

Task 4: $\oplus$
Show that $(A, B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ is controllable if and only if the following holds:
For every $x_0, x_1 \in \mathbb{C}^n$ and every $\tau > 0$ there exists a $u \in \mathcal{C}_\infty^m$ such that the unique solution $x \in \mathcal{C}_\infty^n$ of the initial value problem

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{cases}$$

satisfies $x(\tau) = x_1$.

Task 5: $\circledcirc$

Let $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,m}$. Show that for the (Kalman) controllability matrix it holds

$$\operatorname{image}(K(A,B)) = \operatorname{image}(K(-A,B)) = \operatorname{image}(K(A,-B)) = \operatorname{image}(K(-A,-B)).$$

Conclude that (A,B) is controllable if and only if $(-A,B)$ is controllable if and only if $(A,-B)$ is controllable if and only if $(-A,-B)$ is controllable.

Task 6: $\circledcirc$

In linear time-invariant state-space system with output equation

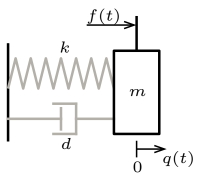
$$\dot{x} \quad = \quad Ax + Bu, \qquad\qquad y = Cx + Du.$$

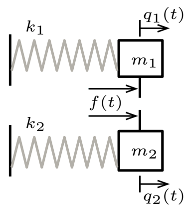
consider x a latent variable and (y,u) as manifest variables. Give a latent variable description $\mathcal{B}$ of the system. Then, let $V \in \mathbb{C}^{n,n}$ be invertible and consider the system

$$\dot{z} = VAV^{-1}z + VBu, \qquad\qquad y = CV^{-1}z + Du$$

where it helps to think of $z = Vx$. Also give a latent variable description $\tilde{\mathcal{B}}$ of this system. Finally, show that $\mathcal{B} = \tilde{\mathcal{B}}$, i.e., by performing a Kalman-decomposition of (A,B) we do not change the manifest behavior of the system (although the latent variable of course changes its form).

Task 7: $\oplus$





(1)

Consider the left mass-spring-damper system in (1) where $q \in \mathcal{C}^1_\infty$ describe the horizontal displacement of the mass over time, $f \in \mathcal{C}^1_\infty$ describes externally applied forces, $k \geq 0$ is the stiffness coefficient of the spring (it describes how stiff the spring is), and $d \geq 0$ is the damping coefficient of the damper (it describes how hard the damper is).

By Newton's second law (which states that mass times acceleration equals force) it makes sense to describe the motion of the mass by

$$m\ddot{q}(t) = -d\dot{q}(t) - kq(t) + f,$$

since the velocity of the mass is $\dot{q}(t)$ and this velocity determines the damping force (the higher the velocity, the higher the damping force; we assume a linear damper). Similar, the second term describes the force from the spring (Hooke's law).

1. Introduce the velocities $v := \dot{q}$ as a new variable and rewrite the system into the form $\dot{x} = Ax + Bu$ with $A \in \mathbb{R}^{2,2}$ and $B \in \mathbb{R}^{2,1}$, where the state contain $x := \begin{bmatrix} q & v \end{bmatrix}^T$ and the input is given by the force $u = f$.

2. Check if the tupel (A,B) from 1.) is controllable, by computing the controllability matrix.

Now consider the right mass-spring-damper system in (1). Since we here assume that there is no damping the equations of motion for the two masses are

$$m_i\ddot{q}_i(t) = -k_i q_i(t) + f,$$

since we can only apply the same force at a certain point in time to both masses.

1. Introduce the velocities $v_i := \dot{q}_i$ for $i = 1,2$ as new variables and rewrite the system into the form $\dot{x} = Ax + Bu$ with $A \in \mathbb{R}^{4,4}$ and $B \in \mathbb{R}^{4,1}$, where the state contain $x := \begin{bmatrix} q_1 & q_2 & v_1 & v_2 \end{bmatrix}^T$ and the input is given by the force $u = f$.

2. Compute the controllability matrix and conclude under for which parameter choices m_1, k_1, m_2, and k_2 the system is controllable.