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Sommersemester 2012

Systems and control theory

Series 6

Task 1:
Implement the staircase algorithm for state-space systems $\dot{x} = Ax + Bu$ with $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,m}$ as described in the proof of Theorem 6 from the handout "Checking controllability numerically" in MATLAB. Then use the implementation to verify the analytically derived result of Series 5, Task 13, part 2 by setting $m_1 = m_2 = 1$ and choosing several values for $d_1, d_2, k_1, k_2 > 0$.
For the implementation, download from the website the zip-file and delete all the code (but not the initial comments and the function declaration) from the file `staircase_form_AB.m`. Then rewrite the code. Thereby, use the supplied function in `compress_rows.m` to make the rank decision. The correctness of the algorithm can be checked by calling

```
>> [err, A, B, corr_n] = multi_test_AB();
```

This function randomly generates and tests multiple examples.

Task 2:
Let $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,m}$. Define $P(\lambda) := \lambda \begin{bmatrix} I & 0 \end{bmatrix} - \begin{bmatrix} A & B \end{bmatrix}$. Show that $\mathcal{B}(P)$ is stabilizable if and only if in the Kalman decomposition of (A, B) the matrix in the (2,2) block A_3 only has eigenvalues with negative real part.

Task 3:
Specify how the staircase algorithm can be used to check stabilizability, observability, and, reconstructability.

Task 4:
Let $U \in \mathbb{C}[\lambda]^{q,m}$ be right prime and consider the system $\mathcal{B} := \operatorname{image}_{\mathcal{C}_\infty}\left(U\left(\frac{d}{dt}\right)\right)$. Show that there exists a left prime polynomial $P \in \mathbb{C}^{p,q}$ such that $\mathcal{B} = \mathcal{B}(P)$.

Task 5:
Complete the proof of Theorem 2.21.

Task 6:
Let $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{p,n}$, and $D \in \mathbb{C}^{p,m}$ and define

$$\tilde{\mathcal{B}} := \left\{ (y, x, u) \in \mathcal{C}^{p+n+m} \middle| \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right\}.$$

Show that $(A, C) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n}$ is observable if and only if the following holds:
For all $(y_1, x_1, u_1), (y_2, x_2, u_2) \in \tilde{\mathcal{B}}$ with $y_1(t) = y_2(t)$ and $u_1(t) = u_2(t)$ for all $t \in [t_0, t_1]$ we have

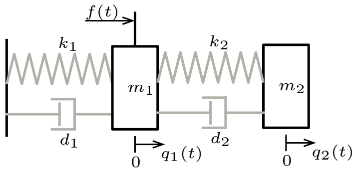
$$x_1(t) = x_2(t), \qquad \text{for all } t \in [t_0, t_1].$$

Task 7:
In Series 3 we considered an electrical circuit with behavior $\mathcal{B}(\lambda F + G)$, where

$$\lambda F + G := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad z := \begin{bmatrix} I_R \\ I_L \\ I \\ V \end{bmatrix}.$$

Is (I_R, I_L, V) observable/reconstructable from I?
Is (I_R, I_L, I) observable/reconstructable from V?
Is (I_R, I, V) observable/reconstructable from I_L?
After answering these questions, apply the staircase algorithm to all cases to determine the canonical from, from which observability/reconstructability can be read off.

Task 8:
Consider the mass-spring-damper system



where $q_1, q_2 \in \mathcal{C}^1_\infty$ describe the horizontal displacement, $f \in \mathcal{C}^1_\infty$ describe externally applied forces, and $k_1, d_1, k_2, d_2 > 0$ are the stiffness and damping coefficients.
By Newton's second law the equation of motion for the second mass is

$$m_2 \ddot{q}_2(t) = d_2 \left(\dot{q}_1(t) - \dot{q}_2(t)\right) + k_2 \left(q_1(t) - q_2(t)\right),$$

since the relative velocity of the second mass against the first mass is $\dot{q}_1(t) - \dot{q}_2(t)$ and this velocity determines the damping force (the higher the relative velocity, the higher the damping force; we assume a linear damper). Similar, the second term describes the force from the spring (Hooke's law). For the first mass we obtain

$$m_1 \ddot{q}_1(t) = -d_1 \dot{q}_1(t) - k_1 q_1(t) - d_2 \left(\dot{q}_1(t) - \dot{q}_2(t)\right) - k_2 \left(q_1(t) - q_2(t)\right) + f(t),$$

where $f \in \mathcal{C}^1_\infty$ is the external force.
One can consider this problem as a model problem for a tuned mass damper (as used to stabilize the motion of skyscrapers). In this case the m_1, d_1, and k_1 are given (they are estimated from the construction of the skyscraper) and one wants to choose the m_2, d_2, and k_2 in such a way that the following is fulfilled: external forces f (like winds or earthquake) which act on the building (i.e., on q_1) lead to small dislocations of the building and m_2 is small compared to m_1.
The behavioral equations in matrix form are then

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} d_1 + d_2 & -d_2 \\ -d_2 & d_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} f$$

1. For which coefficients m_i, d_i, k_i is q_1 observable/reconstructable from (q_2, f)?
2. For which coefficients is q_2 observable/reconstructable from (q_1, f)?
3. Introduce the additional output variable $y = q_i$, with $i = 1, 2$ consider the forces to be input $f =: u$, perform and order reduction, and rewrite the system in the from

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = C_i x(t),$$

where $A \in \mathbb{C}^{4,4}$, $B \in \mathbb{C}^{4,1}$, and $C_i \in \mathbb{C}^{1,4}$. Use the MATLAB staircase implementation from Task 1 to verfiy the analytical results from 1. and 2. numerically.