Technische Universität Berlin **Sommersemester 2012**
Fakultät II – Mathematik und Naturwissenschaften
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Systems and control theory

Series 7

Task 1:
Give the proof of the following proposition from the lecture: Let $R \in \mathbb{C}[\lambda]^{r,q}$ and $P \in \mathbb{C}[\lambda]^{p,q}$. Show that then the following are equivalent:

1. There exists a controller $C \in \mathbb{C}[\lambda]^{s,q}$ such that $\mathcal{B}(R) = \mathcal{B}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$.
2. $\mathcal{B}(R) \subset \mathcal{B}(P)$.

Task 2:
Let $P \in \mathbb{C}[\lambda]^{p,q}$ be such that $\mathcal{B}(P) = \mathcal{C}_\infty^q$. Show that then $P = 0$.
Remark: First show the statement for $p = q = 1$. Use this to show the result for $p \geq 1$ and $q = 1$. Finally, generalize to the most general case.

Task 3:
Let $D \in \mathbb{C}[\lambda]^{r,r}$ be diagonal and let $P \in \mathbb{C}[\lambda]^{p,r}$ be such that

$$\forall \alpha \in \mathcal{C}_\infty^r \text{ with } D\left(\frac{d}{dt}\right)\alpha = 0 \qquad \Rightarrow \qquad P\left(\frac{d}{dt}\right)\alpha = 0.$$

Then there exists a $M \in \mathbb{C}[\lambda]^{p,r}$ such that $MD = P$.
Remark: First show the statement for $p = r = 1$. Use this to show the result for $p \geq 1$ and $r = 1$. Finally, generalize to the most general case.

Task 4:
Let $D \in \mathbb{C}[\lambda]^{r,q}$ have full row rank (over $\mathbb{C}(\lambda)$) and let $M \in \mathbb{C}[\lambda]^{p,r}$. Show that

$$\mathcal{B}(MD) = \mathcal{B}(D) \qquad \Leftrightarrow \qquad \mathcal{B}(M) = \{0\}.$$

Remark: For "$\Rightarrow$" use Lemma 1.17 a).

Task 5:
Show that the assumption of controllability in Theorem 3.3 is really necessary. More precisely, show that there exists a $P \in \mathbb{C}[\lambda]^{p,q}$ for which $\mathcal{B}(P)$ is not controllable and an $R \in \mathbb{C}[\lambda]^{r,q}$ with $\mathcal{B}(R) \subset \mathcal{B}(P)$ such that for every $C \in \mathbb{C}[\lambda]^{c,q}$ with

$$\mathcal{B}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = \mathcal{B}(R),$$

we have $\operatorname{rank}_{\mathbb{C}(\lambda)}(P) + \operatorname{rank}_{\mathbb{C}(\lambda)}(C) > \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$.
Remark: One can choose $p = q = 1$.

Task 6:
Give the details of the proof of Corollary 3.5., i.e., show that for stabilizable $\mathcal{B}(P)$ with $P \in \mathbb{C}[\lambda]^{p,q}$ there exists a regular, stabilizable, left prime controller which stabilizes the system.

Task 7:
Assume that $C_1 \in \mathbb{C}[\lambda]^{c_1,q_1}$ is a regular (and stabilizing) controller for $P_1 \in \mathbb{C}[\lambda]^{p_1,q_1}$ and $C_2 \in \mathbb{C}[\lambda]^{c_2,q_2}$ is a regular (and stabilizing) controller for $P_2 \in \mathbb{C}[\lambda]^{p_2,q_2}$.
Show that then $\begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix}$ is a regular (and stabilizing) controller for $\begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}$.

Task 8:
Assume that $P \in \mathbb{C}[\lambda]^{p,q}$ has full column rank. Show that then there exists no regular controller (with at least one row) for $\mathcal{B}(P)$.

Task 9:
Let $P_1, P_2 \in \mathbb{C}[\lambda]^{p,q}$ be with $\mathcal{B}(P_1) = \mathcal{B}(P_2)$. Show that then also the zeros are the same, i.e., we have $\mathfrak{Z}(P_1) = \mathfrak{Z}(P_2)$.

Task 10:
Consider the familiar electric circuit with behavior $\mathcal{B}(\lambda F + G)$, where

$$\lambda F + G(\lambda) := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad z := \begin{bmatrix} I_R \\ I_L \\ I \\ V \end{bmatrix}.$$

1. Assume that we attach a capacitor across the external wires, i.e., assume we impose the additional constraint $C\dot{V} = I$ with $C > 0$. Is such a capacitor a regular controller? Is it stabilizing?
2. Repeat 2.) for the case that a resistor $V = \tilde{R}I$ or a inductor $\tilde{L}\dot{I} = V$ is attached across the external wires.
3. Determine polynomials $c_I, c_V \in \mathbb{C}[\lambda]$ such that the controller $c_I\left(\frac{d}{dt}\right) I = c_V\left(\frac{d}{dt}\right) V$ is **not** regular

Task 11:
Let $m, d, k \in \mathbb{R}_{>0}$ and consider the mass-spring-damper system

$$\mathcal{B}\left(\begin{bmatrix} \lambda^2 m + \lambda d + k & -1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} q \\ f \end{bmatrix} \in \mathcal{C}^2_\infty \middle| m\ddot{q} + d\dot{q} + k = f \right\}.$$

Assume that we can measure the velocity of the mass $\dot{q}$ and we want to use this information to set the external force f. Since here we only deal with linear equations/controllers we make the ansatz $c\dot{q} = u$ where $c \in \mathbb{R}$ can also be called the *gain* of the controller. This leads to the system $\mathcal{B}(P)$ with

$$P(\lambda) := \begin{bmatrix} \lambda^2 m + \lambda d + k & -1 \\ \lambda c & -1 \end{bmatrix}.$$

1.) For which c is this controller regular? For which c is the resulting system autonomous?

For autonomous systems all solutions have the form $\sum_{\lambda \in \mathfrak{Z}(P)} p_\lambda(t) e^{\lambda t}$ where for each $\lambda \in \mathfrak{Z}(P)$ the functions p_λ are polynomials. If one now concentrates on the exponential terms we see that $e^{\lambda t} \overset{t\to\infty}{\to} 0$ if and only if $\mathrm{Re}(\lambda) < 0$ (cf. Lemma 3.3) an the convergence is quicker, the smaller the real part of λ is.

2.) Compute the gain c which is optimal in the sense that the maximum real part of a zeros of P is as small as possible.