Technische Universität Berlin **Sommersemester 2012**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Systems and control theory

Series 8

Task 1:
Let $\epsilon \in \mathbb{N}_0$ and $c_0, \dots, c_\epsilon \in \mathbb{C}$. Let $\mathcal{L}_\epsilon \in \mathbb{C}[\lambda]_1^{\epsilon,\epsilon+1}$ be a block in the Kronecker canonical form of type $\mathcal{L}$ and size ϵ, i.e., let

$$\mathcal{L}(\lambda) := \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix},$$

and set $C := \begin{bmatrix} c_0 & \cdots & c_\epsilon \end{bmatrix} \in \mathbb{C}^{1,\epsilon+1}$ and $p(\lambda) := \lambda^\epsilon c_\epsilon + \ldots + \lambda c_1 + c_0$. Show that then

$$\mathfrak{Z}\left(\begin{bmatrix} \mathcal{L}(\lambda) \\ C \end{bmatrix}\right) = \mathfrak{Z}\left(p(\lambda)\right), \qquad \text{and} \qquad \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \mathcal{L}(\lambda) \\ C \end{bmatrix}\right) = \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\mathcal{L}\right) + \operatorname{rank}_{\mathbb{C}(\lambda)}\left(p(\lambda)\right).$$

In other words, this shows that C is a regular controller for $\mathcal{B}(\mathcal{L})$ which makes the controlled system autonomous if and only if $C \neq 0$ and the zeros of the controlled system can be chosen freely as the zeros of the polynomial p.
Hint: Remember Lemma 4 from the handout "First order systems".

Task 2:
Let $C_2 \in \mathbb{C}^{m,m}$ be invertible (over $\mathbb{C}$ = over $\mathbb{C}(\lambda)$; since C_2 is constant). Let $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, and $C_1 \in \mathbb{C}^{m,n}$.
(a) Show that $P(\lambda) := \begin{bmatrix} \lambda I - A & -B \\ -C_1 & -C_2 \end{bmatrix} \in \mathbb{C}(\lambda)^{n+m,n+m}$ is invertible (over $\mathbb{C}(\lambda)$).
(b) Conclude that $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ is a regular controller for $\begin{bmatrix} \lambda I - A & -B \end{bmatrix}$.
(c) Construct A, B, C_1, C_2 such that $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ is a regular controller for $\begin{bmatrix} \lambda I - A & -B \end{bmatrix}$ but C_2 is not invertible (over $\mathbb{C}$).

Task 3:
Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $R \in \mathbb{C}[\lambda]^{r,q}$. Show that $\mathcal{B}\left(\begin{bmatrix} P & 0 \\ -R & I \end{bmatrix}\right) = \begin{bmatrix} I \\ R\left(\frac{d}{dt}\right) \end{bmatrix} \mathcal{B}(P)$.

Task 4:
Let $R_{11} \in \mathbb{C}[\lambda]^{p_1,q_1}$, $R_{12} \in \mathbb{C}[\lambda]^{p_1,q_2}$, and $R_{22} \in \mathbb{C}[\lambda]^{p_2,q_2}$. In this task conditions shall be determined under which we have

$$\mathfrak{Z}\left(\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}\right) = \mathfrak{Z}\left(R_{11}\right) \cup \mathfrak{Z}\left(R_{22}\right). \tag{1}$$

(a) If R_{11} has full row rank and if R_{22} has full column rank then (1) holds.
(b) Construct an R_{11} which has full column rank and an R_{22} which has full row rank but (1) does not hold.
Hint: For (b) one can choose $R_{11} \in \mathbb{C}[\lambda]^{2,1}$ and $R_{22} \in \mathbb{C}[\lambda]^{1,2}$.

Task 5:
For a block of type $\mathcal{L} \in \mathbb{C}[\lambda]^{\epsilon,\epsilon+1}$ in the Kronecker canonical form give a constant regular stabilizing controller $C \in \mathbb{C}^{1,\epsilon+1}$ such that $\mathfrak{Z}\left(\begin{bmatrix} \mathcal{L}(\lambda) \\ C \end{bmatrix}\right) = \emptyset$ and $\mathcal{B}\left(\begin{bmatrix} \mathcal{L}(\lambda) \\ C \end{bmatrix}\right) = \{0\}$.

Task 6:
Give the polynomial matrix $P \in \mathbb{C}[\lambda]_1^{p,q}$ such that $\mathcal{B}(P)$ describes the interaction of the system (S) and the observer (O) from the section "Reconstruction via observer-synthesis" such that the variables appear in the order $z = \begin{bmatrix} y^T & x^T & \hat{y}^T & \hat{x}^T & u^T \end{bmatrix}^T$.

Task 7:

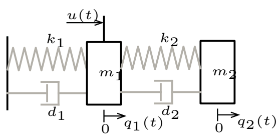
1. Show that the problem of *stabilization via state-feedback* (cf. Corollary 3.6) also has a solution under the weaker assumption that $\mathcal{B}\left(\begin{bmatrix} \lambda I - A & -B \end{bmatrix}\right)$ is stabilizable.
2. Show that the problem of *reconstruction via observer-synthesis* (cf. Corollary 3.7) also has a solution under the weaker assumption that $\mathcal{B}\left(\begin{bmatrix} \lambda I - A \\ -C \end{bmatrix}\right)$ is reconstructable.
3. Show that the problem of finding a *compensator* (cf. Corollary 3.8) also has a solution under the weaker assumption that $\mathcal{B}\left(\begin{bmatrix} \lambda I - A & -B \end{bmatrix}\right)$ is stabilizable and $\mathcal{B}\left(\begin{bmatrix} \lambda I - A \\ -C \end{bmatrix}\right)$ is reconstructable.

Task 8:
In Series 3, Task 8 we saw that the Kronecker canonical form for the familiar electirc circuit is given by

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ R & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=:S^{-1}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & L & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & R & R & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}_{=:T}.$$

Use this and the construction from the handout "Constant controllers for first order systems" to construct a constant regular stabilizing controller such that the controlled system has one zero at $d \in \mathbb{R}$.

Task 9:
In Series 6, Task 8, 3. we saw that the mass-spring-damper system



with unit masses $m_1 = m_2 = 1$ is described by a state-space system $\dot{x} = Ax + Bu$, $y = Cx$ where

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 - k_2 & k_2 & -d_1 - d_2 & d_2 \\ k_2 & -k_2 & d_2 & -d_2 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C := \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thereby, the choice of C reflected that it is only possible to observe the position of the first mass, but not the position of the second. The motivation of this task is to synthesize an observer which reveals the position of the second mass.
If we have $k_1, k_2, d_1, d_2 > 0$ it seems to make physical sense (although a proof is not trivial) that the system is stable, i.e., we have $\sigma(A) \subset \mathbb{C}_-$. Thus, to synthesize an observer, we can simply choose $L := 0$ to obtain $\sigma(A + LC) \subset \mathbb{C}_-$.
Things get more interesting, when $d_1 = d_2 = 0$, since in this case A is not stable. Show the following:

1. If $K = K^* \in \mathbb{C}^{n,n}$ is positive semi-definite then all eigenvalues of $\begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}$ are purely imaginary.
2. Conclude that all eigenvalues of A are purely imaginary, if $k_1, k_2 \geq 0$ and $d_1 = d_2 = 0$.
3. In Series 6, Task 8. we saw that (A, C) is observable. Thus Corollary 3.7 implies that there exists an $L \in \mathbb{C}^{4,1}$ such that $\sigma(A + LC) \subset \mathbb{C}_-$. Use MATLAB in an trial-and-error manner to determine such an L for the case that $k_1 = k_2 = 1$ and $d_1 = d_2 = 0$. In the same manner show that not every L is appropriate.