Technische Universität Berlin **Sommersemester 2012**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
Tobias Brüll

Systems and control theory

Series 9

Task 1:
Show that with $R \in \mathbb{C}(\lambda)^{p,q}$ the para-conjugate transposed $R^{\sim}$ is indeed again a rational matrix. Then show that for $A \in \mathbb{C}(\lambda)^{p,p}$, $B \in \mathbb{C}(\lambda)^{p,q}$, $C \in \mathbb{C}(\lambda)^{q,r}$, and polynomial $U \in \mathbb{C}[\lambda]^{p,p}$ we have

1.) $(BC)^{\sim} = C^{\sim}B^{\sim}$
2.) $\left(A^{-1}\right)^{\sim} = \left(A^{\sim}\right)^{-1}$, if A is invertible
3.) $(B^{\sim})^{\sim} = B$
4.) With A also $B^{\sim}AB$ is para-Hermitian
5.) With U also $U^{\sim}$ is unimodular
6.) $\operatorname{rank}_{\mathbb{C}(\lambda)}(B) = \operatorname{rank}_{\mathbb{C}(\lambda)}(B^{\sim})$
7.) If B has full row rank then $B^{\sim}$ has full column rank

Task 2:
For a polynomial matrix of the form $P(\lambda) = \sum_{i=0}^{K} \lambda^i P_i$ give the para-conjugate transposed in the same form $P^{\sim}(\lambda) = \sum_{i=0}^{K} \lambda^i \ldots$.

Task 3:
Show that for real systems it is not a restriction to assume that $H = H^*$. In other words, let $P \in \mathbb{R}[\lambda]^{p,q}$ be a polynomial with real coefficients. Assume that we are only interested in the real trajectories $z \in \mathcal{B}(P)$, i.e., the trajectories of the form $z : \mathbb{R} \to \mathbb{R}^q$. Further assume that we measure the cost (or power supply) at time point $t \in \mathbb{R}$ via $(\Delta_K z(t))^* M(\Delta_K z(t))$ where $M \in \mathbb{R}^{Kq,Kq}$ is real but has no further structure. Give an $H = H^* = H^T \in \mathbb{R}^{Kq,Kq}$ such that $(\Delta_K z(t))^* M(\Delta_K z(t)) = (\Delta_K z(t))^* H(\Delta z(t))$ for all real $z \in \mathcal{B}(P)$ and all $t \in \mathbb{R}$.

Task 4: Part 1. is not easy.

1. Let $a \in \mathbb{C}$. Show that for every $f \in \mathcal{C}^1_+$ there exists a $x \in \mathcal{C}^1_+$ such that $\dot{x} = ax + f$ by using the variation of constants formula.
2. Let $A \in \mathbb{C}^{n,n}$. Show that for every $f \in \mathcal{C}^n_+$ there exists a $x \in \mathcal{C}^n_+$ such that $\dot{x} = Ax + f$ by using the Jordan canonical form and point 1.
3. Let $d \in \mathbb{C}[\lambda]$, $d \neq 0$. Show that for every $b \in \mathcal{C}^1_+$ there exists a $z \in \mathcal{C}^q_+$ such that $d\left(\frac{d}{dt}\right) z = b$ by replacing the higher-order, scalar differential equation by a first-order equation and using point 2.
4. Let $P \in \mathbb{C}[\lambda]^{p,q}$ have full row rank. Show that for every $f \in \mathcal{C}^p_+$ there exists a $z \in \mathcal{C}^q_+$ such that $P\left(\frac{d}{dt}\right) z = b$ by using the Smith canonical form and point 3.

Task 5:
Let $P \in \mathbb{C}[\lambda]^{p,q}$ have full column rank with $\mathfrak{Z}(P) \subset \mathbb{C}_-$. Show that then the optimal control problem (LQ) has a unique solution and that every $H = H^* \in \mathbb{C}^{Kq,Kq}$ is non-negative w.r.t to P.

Task 6:
Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Assume that H is **not** non-negative w.r.t. P. Let $z_0 \in \mathcal{B}_+(P)$. Show that it is then possible to construct trajectories of arbitrary low cost, which are equal to z_0 on $(-\infty, 0]$. This especially implies that the optimal control problem is unsolvable.
More precisely, show the following: For every $M \in \mathbb{R}$ there exists a $\hat{z} \in \mathcal{B}_+(P)$ such that $z_0(t) = \hat{z}(t)$ for all $t \leq 0$ and

$$\int_0^\infty (\Delta_K \hat{z}(t))^* H(\Delta_K \hat{z}(t)) dt < M.$$

Hint: Consider $z_0 + \alpha v$ where $\alpha \in \mathbb{R}$ and $v \in \mathcal{B}_+(P)$ violates the assumption of non-negativity.

Task 7:

Derive the partial integration rule $\int_{t_0}^{t_1} z(t)\dot{y}(t)dt = z(t)y(t)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{z}(t)y(t)dt$ from Lemma 4.2.

Remark: The partial integration rule was used in the proof of Lemma 4.2.

Task 8:

Let $P \in \mathbb{C}[\lambda]^{p,q}$. Show that (a) with $z \in \mathcal{C}_+^q$ also $P\left(\frac{d}{dt}\right) z \in \mathcal{C}_+^p$ is exponentially decaying and (b) if $z \in \mathcal{C}_\infty^q$ fulfills $z(t) = 0$ for $t \leq 0$ then also $P\left(\frac{d}{dt}\right) z(t) = 0$ for $t \leq 0$.

Task 9:

Let $m, d, k > 0$. Consider the mass-spring-damper system described by

$$m\ddot{q}(t) + d\dot{q}(t) + kq(t) = f(t),$$

where $q, f \in \mathcal{C}_\infty^1$. Physicists in general agree that the power supply should be measured as

$$\text{velocity of the mass} \times \text{applied force} = \dot{q} \cdot f,$$

and that only real trajectories $q, f : \mathbb{R} \to \mathbb{R}$ are of interest. This especially means, that (a) if we apply force in the direction of the velocity of the mass, then we supply (a positive amount of) energy to the system and (b) if we apply force in the opposite direction of the velocity of the mass, then we extract energy from the system (i.e., we supply a negative amount of energy to the system)

Let the trajectory $(q_0, f_0) \in \mathcal{B}_+(\begin{bmatrix}\lambda^2 m + \lambda d + k & -1\end{bmatrix}) =: \mathcal{B}_+(P(\lambda))$ describe the motion of the system in the past $t \leq 0$. Rewrite each of the following problems as an optimal control problem (LQ). This mainly means that you have to choose the correct $H = H^*$. Then also give the optimality system.

1.) Assume that we want to extract as much energy from the system as possible, i.e., we want to solve

$$\sup_{\substack{(q,f) \in \mathcal{B}_+(P) \\ (q(t),f(t)) = (q_0(t), f_0(t)), t \leq 0}} -\int_0^\infty \mathrm{Re}\left(\dot{q}(t)\overline{f(t)}\right) dt.$$

Here you have to use that we are only interested in real trajectories, also see Task 3.

2.) Assume that we want to minimize the total motion/dislocation of the mass, i.e., we want to solve

$$\inf_{(q,f) \in \dots} \int_0^\infty |\dot{q}(t)|^2 + |q(t)|^2 dt.$$

3.) Assume that we want to minimize the total motion and dislocation of the mass but also want to penalize the amount of force which has to be used, i.e., we want to solve the so-called *regularized* problem

$$\inf_{(q,f) \in \dots} \int_0^\infty \frac{|\dot{q}(t)|^2}{2} + \frac{|q(t)|^2}{2} + \frac{\epsilon}{2}|f(t)|^2 dt,$$

where $\epsilon > 0$.

Task 10:

Consider the familiar electrical circuit

$$P(\lambda) := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad z := \begin{bmatrix} I_R \\ I_L \\ I \\ V \end{bmatrix},$$

where $z \in \mathcal{C}_\infty^4$. Physicists in general agree that the supplied energy should be measured as

$$\text{current} \times \text{voltage} = I \cdot V,$$

if only real trajectories $z : \mathbb{R} \to \mathbb{R}^4$ are considered. Assume we know that the system evolved until the time point $t = 0$ along the trajectory $z_0 \in \mathcal{B}_+(P)$ and we want to extract as much energy from the system as possible, i.e., we want to maximize

$$\sup_{\substack{z \in \mathcal{B}_+(P) \\ z(t) = z_0(t), t \leq 0}} -\int_0^\infty \mathrm{Re}\left(I(t)\overline{V(t)}\right) dt.$$

Rewrite the problem as an optimal control problem of the form (LQ). This mainly means that you have to choose the correct $H = H^*$. Then also give the optimality system.