

**Systems and control theory**  
**Series 11**

For this complete series we use the notation

$$J := J_p := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and with  $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$  and  $H = H^* \in \mathbb{C}^q$  we denote by

$$\mathfrak{N}(\lambda) := \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q,p+q},$$

the matrix of the optimality system.

**Task 1:**

Show that under Assumptions 4.8 the matrix polynomial  $\mathfrak{N}$  is invertible.

**Task 2:**

Let  $Y \in \mathbb{C}^{q,q}$ . Show that then there exists a unitary matrix  $W \in \mathbb{C}^{q,q}$  such that  $YW$  is Hermitian.

*Hint:* Use the singular value decomposition.

**Task 3:**

Show that

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & C^* C \end{bmatrix} \quad (1)$$

is equivalent to

$$Z^* F = F^* Z \text{ and } H + G^* Z + Z^* G = C^* C.$$

**Task 4:**

Let  $N = N^* \in \mathbb{C}[\lambda]^{p,p}$  be a para-Hermitian polynomial matrix. Show that  $N$  is Hermitian on the imaginary axis, i.e., show that  $N(i\omega_0) = (N(i\omega_0))^* \stackrel{\text{def}}{=} N^*(i\omega_0)$  for all  $\omega_0 \in \mathbb{R}$ .

**Task 5:** (The energy perspective)

Let  $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ ,  $H = H^* \in \mathbb{C}^{q,q}$ , and let  $Z, C$  fulfill (1). Define the quadratic function  $\Theta : \mathbb{C}^q \rightarrow \mathbb{R}$  via

$$\Theta(y) := y^* F^* Z y = y^* Z^* F y.$$

Show that  $\Theta$  is a so-called *storage function*, i.e., that it fulfills the so-called *dissipation inequality*

$$\Theta(z(t_1)) - \Theta(z(t_0)) \leq \int_{t_0}^{t_1} z^*(t) H z(t) dt, \quad (2)$$

for all  $t_0 < t_1$  and all  $z \in \mathcal{B}(\lambda F + G)$ .

Conclude that then  $H$  is non-negative w.r.t.  $\lambda F + G$ .

*Hint:* Use the fundamental theorem  $\Theta(z(t_1)) - \Theta(z(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt} (\Theta(z(t))) dt$  and Task 3.

*Remark:* One can think of  $\Theta$  as measuring the energy which is stored internally in the system  $\mathcal{B}(\lambda F + G)$ . The term on the left side of (2) then measures the gain in internally stored energy, while the term on the right side of (2) measures the energy supply in the time interval  $[t_0, t_1]$ .

**Task 6:** (The cost perspective)

Let  $P(\lambda) := \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ ,  $H = H^* \in \mathbb{C}^{q,q}$ , and let  $Z, C$  fulfill (1). Define the quadratic function  $\Theta : \mathbb{C}^q \rightarrow \mathbb{R}$  via

$$\Theta(y) := -y^* F^* Z y = -y^* Z^* F y.$$

Show that  $\Theta$  measures the cost of an optimal trajectory, i.e., show that for any trajectory  $\hat{z} \in \mathcal{B}_+(P)$  which is optimal in the sense of (LQ) (for some  $z_0 \in \mathcal{B}_+(\lambda F + G)$ ) we have

$$\Theta(\hat{z}(0)) = \int_0^\infty \hat{z}^*(t) H \hat{z}(t) dt.$$

*Hint:* Use Theorem 4.10, 2.) b).

*Remark:* Compare this with Theorem 4.5.

**Task 7:**

A matrix  $\mathcal{H} \in \mathbb{C}^{2n,2n}$  is called *Hamiltonian* if  $\mathcal{H}J$  is Hermitian.

A matrix  $\mathcal{S} \in \mathbb{C}^{2n,2n}$  is called *symplectic* if  $\mathcal{S}^* J \mathcal{S} = J$ .

(a) Use  $J^{-1} = -J = J^*$  to show that the following are equivalent:

1.  $\mathcal{H}$  is Hamiltonian.
2.  $\mathcal{H}J = -J\mathcal{H}^*$  is Hamiltonian.
3.  $J\mathcal{H}$  is Hermitian.

4. There exist matrices  $A = A^*, B, C = C^* \in \mathbb{C}^{n,n}$  such that  $\mathcal{H} = \begin{bmatrix} B & A \\ C & -B^* \end{bmatrix}$ .

(b) Show that for the eigenvalues of a Hamiltonian matrix  $\mathcal{H} \in \mathbb{C}^{2n,2n}$  we have

$$\sigma(\mathcal{H}) = \mathfrak{Z}(\lambda J - (\mathcal{H}J)),$$

and conclude that the eigenvalues of a Hamiltonian matrix are symmetric w.r.t. the imaginary axis.

(c) (Hamiltonian Schur form) Let  $\mathcal{H} \in \mathbb{C}^{2n,2n}$  be a Hamiltonian matrix which has no purely imaginary eigenvalues. Show that in this case there exists a symplectic unitary transformation  $\mathcal{S} \in \mathbb{C}^{2n,2n}$  which brings  $\mathcal{H}$  to the Hamiltonian Schur form

$$\mathcal{S}^* \mathcal{H} \mathcal{S} = \begin{bmatrix} L & K \\ 0 & -L^* \end{bmatrix},$$

where  $K = K^* \in \mathbb{C}^{n,n}$  is Hermitian and  $L \in \mathbb{C}^{n,n}$  is a lower triangular matrix which only has eigenvalues with negative real part.

*Remark:* Control theory courses are often based on Hamiltonian matrices, rather than para-Hermitian matrix polynomials.

**Task 8:**

For the system

$$\dot{x}(t) = u(t),$$

where  $x, u \in \mathcal{C}_\infty^1$ , consider the optimal control problem

$$\inf_{x(0)=x_0} \int_0^\infty |x(t)|^2 + |u(t)|^2 dt. \quad (3)$$

Compute  $c_1, c_2 \in \mathbb{C}$  such that the solution of the controlled system

$$\begin{cases} \dot{x}(t) &= u(t) \\ 0 &= c_1 x(t) + c_2 u(t) \end{cases},$$

solve the optimal control problem (3). Also give the closed-loop dynamics.

*Hint:* Follow the construction in the proof of Theorem 4.10. Note that the  $W$  was only needed to show that  $Y_4 = \tilde{Y}_4 W$  is invertible. Here one can immediately see that  $\tilde{Y}_4$  is already invertible. Thus (in the proof of Theorem 4.10) one can set  $Y_4 := \tilde{Y}_4$ ,  $Y_2 := \tilde{Y}_2$  and with this immediately construct  $\hat{Y}$  to obtain the  $Z$ . Finally, use the equivalence in (1) to compute  $C = [c_1 \quad c_2]$  as a factor of  $H + G^* Z + Z^* G = C^* C$ .