Series 1: Solutions

Task 1:

<u>kernel</u>: Let $P \in \mathbb{C}[\lambda]^{p,q}$, and $z_1, z_2 \in \mathfrak{B}(P)$.
Let $\alpha \in \mathbb{C}$.

$$\Rightarrow \quad P(\tfrac{d}{dt}) z_1 = 0 \,, \quad P(\tfrac{d}{dt}) z_2 = 0$$

$$\Rightarrow \quad \alpha \; P(\tfrac{d}{dt}) z_1 = 0$$

$$\Rightarrow \quad P(\tfrac{d}{dt})(\alpha z_1 + z_2) = \alpha P(\tfrac{d}{dt}) z_1 + P(\tfrac{d}{dt}) z_2 = 0$$

$$\Rightarrow \quad \alpha z_1 + z \in \mathfrak{B}(P) \quad \Rightarrow \text{linear}$$

Furthermore, for $z \in \mathfrak{B}(P)$, $\tau \in \mathbb{R}$
we have

$$P(\tfrac{d}{dt}) z(t) = 0 \qquad \forall t \in \mathbb{R}$$

$$\Rightarrow \quad P(\tfrac{d}{dt}) z(t+\tau) \qquad \forall t \in \mathbb{R}$$

$$\Rightarrow \quad z(\cdot + \tau) \in \mathfrak{B}(P) \quad \Rightarrow \text{time invariance}$$

<u>image</u>: Let $U \in \mathbb{C}[\lambda]^{q,m}$ and

$z_1, z_2 \in \operatorname{image}_{\mathcal{C}_\infty} U$, i.e., there
exist $\alpha_1, \alpha_2 \in \mathcal{C}_\infty^m$ s.t.

$$z_1 = U(\tfrac{d}{dt})\alpha_1 \,, \quad z_2 = U(\tfrac{d}{dt})\alpha_2$$

$$\Rightarrow \quad \beta z_1 + z_2 = U(\tfrac{d}{dt}) \underbrace{(\beta \alpha_1 + \alpha_2)}_{\in \mathcal{C}_\infty^m}$$

$$\Rightarrow \quad \beta z_1 + z_2 \in \operatorname{image}_{\mathcal{C}_\infty} U \quad \Rightarrow \text{linear}$$

Furthermore, for $z \in \text{image}_{\mathcal{C}_\infty} U$ there exists $\alpha \in \mathcal{C}_\infty^m$ such that

$$z(t) = U(\tfrac{d}{dt})\,\alpha(t) \qquad \forall t \in \mathbb{R}$$

$$\Rightarrow\ z(t+\tau) = U(\tfrac{d}{dt})\,\underbrace{\alpha(t+\tau)}_{=:\,\tilde{\alpha}(t)} = U(\tfrac{d}{dt})\,\tilde{\alpha}(t)$$

$$\Rightarrow\ z(\cdot+\tau) \in \text{image}_{\mathcal{C}_\infty} U.$$

$\Rightarrow$ time invariance

2.) For $(y,x,u) = \begin{bmatrix} y \\ x \\ u \end{bmatrix} \in \mathcal{C}_\infty^{\ell+n+m}$

we have

$$(*)\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

is equivalent to

$$0 = \begin{bmatrix} \frac{d}{dt}x - Ax - Bu \\ y - Cx - Du \end{bmatrix} = \begin{bmatrix} 0 + (\frac{d}{dt}I - A)x & -Bu \\ y - \quad Cx & -Du \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{d}{dt}I - A & -B \\ I & -C & -D \end{bmatrix} \begin{bmatrix} y \\ x \\ u \end{bmatrix}.$$

Thus with $P(\lambda) := \begin{bmatrix} 0 & \lambda I - A & -B \\ I & -C & -D \end{bmatrix}$ ~~$\in \mathbb{C}[\lambda]^{n+\ell}$~~ $\in \mathbb{C}[\lambda]^{n+\ell,\,\ell+n+m}$

~~$(*)$~~ and $z = \begin{bmatrix} y \\ x \\ u \end{bmatrix}$ ~~$(*)$ is equiva~~

(*) is equivalent to $P(\frac{d}{dt})z = 0$.

3.) As before set

$$P(\lambda) := [\lambda^2 M + \lambda D + K \ , \ -B]$$

and partition $z = \begin{bmatrix} x \\ u \end{bmatrix}$ so that the system is equivalent to

$$P(\tfrac{d}{dt}) z = 0.$$

Task 2:

A matrix Q is unimodular if and only if

$$\det Q \equiv c \neq 0 \quad , \quad c \in \mathbb{C}$$

(see Theorem from lecture).

If P is also unimodular with matching dimension we find

$$\det(PQ) = \underbrace{\det P}_{= d \in \mathbb{C}\setminus\{0\}} \cdot (\det Q) = d \cdot c \in \mathbb{C},$$

i.e., PQ is unimodular.

Task 4:

1.) If $z \in \mathcal{C}_\bullet^q$ then so are

$z, z^{(1)}, \dots, z^{(k)} \in \mathcal{C}_\bullet^q$. Thus also for $P_k, \dots, P_0 \in \mathbb{C}^{p,q}$

$$P_k z^{(k)}, \dots, P_1 z^{(1)}, P_0 z \in \mathcal{C}_\bullet^p$$

$$\Rightarrow P_k z^{(k)} + \dots + P_1 z^{(1)} + P_0 z \in \mathcal{C}_\bullet^p$$

Thus for $P(\lambda) = \lambda^k P_k + \dots + \lambda P_1 + P_0$
we have

$$P(\tfrac{d}{dt}) z = (\tfrac{d}{dt})^k P_k z + \dots + (\tfrac{d}{dt}) P_1 z + P_0 z$$

$$= P_k z^{(k)} + \dots + P_1 z_1^{(1)} + P_0 z \in \mathcal{C}_\bullet^p$$

2.) Let

$$P(\lambda) = \sum_{i=0}^{K} P_i \lambda^i, \quad P_i \in \mathbb{C}^{p \times q}$$

$$Q(\lambda) = \sum_{i=0}^{K} Q_i \lambda^i, \quad Q_i \in \mathbb{C}^{q \times m}.$$

where some P_i, Q_i may be zero.
~~where~~ $\Rightarrow (PQ)(\lambda) = \sum_{i=0}^{2K} \lambda^i \sum_{j=0}^{i} P_{i-j} Q_j$.

$$\Rightarrow P(\tfrac{d}{dt}) Q(\tfrac{d}{dt}) z = P(\tfrac{d}{dt}) \sum_{i=0}^{K} Q_i z^{(i)}$$

$$= \sum_{k=0}^{K} P_k (\tfrac{d}{dt})^k \cancel{Q_i z} \sum_{i=0}^{K} Q_i z^{(i)}$$

$$= \sum_{k=0}^{K} \sum_{i=0}^{K} P_k Q_i z^{(i+k)}$$

2.) Let
$$P(\lambda) = \sum_{i=0}^{K} \lambda^i P_i$$
$$Q(\lambda) = \sum_{i=0}^{K} \lambda^i Q_i$$
where some of the P_i ~~Q_i~~ $\in \mathbb{C}^{p,q}$ $Q_i \in \mathbb{C}^{q,m}$ may be zero. Then

$$\begin{aligned}(PQ)(\lambda) &= P(\lambda)\cdot Q(\lambda)\\ &= \left(\sum_{i=0}^{K} \lambda^i P_i\right)\cdot\left(\sum_{j=0}^{K} \lambda^j Q_j\right)\\ &= \sum_{i,j=0}^{K} \lambda^{i+j} P_i Q_j\end{aligned}$$

and thus

$$\begin{aligned}P(\tfrac{d}{dt})\,Q(\tfrac{d}{dt})\,z &= P(\tfrac{d}{dt})\sum_{i=0}^{K} Q_i z^{(i)}\\ &= \sum_{j=0}^{K} P_j \left(\tfrac{d}{dt}\right)^j \sum_{i=0}^{K} Q_i z^{(i)}\\ &= \sum_{j=0}^{K}\sum_{i=0}^{K} P_j Q_i z^{(i+j)} = (PQ)(\tfrac{d}{dt})\,z\end{aligned}$$

3.) Follows from 2 since for unimodular matrices U we have

$$U^{-1}(\lambda)\,U(\lambda) = (U^{-1}U)(\lambda) = I.$$

Task 5: 1.) We have

$$\mathfrak{B}(SPT) = \left\{ z \in \mathcal{C}_\infty^q \;\middle|\; \underbrace{S(\tfrac{d}{dt})\, P(\tfrac{d}{dt})\, T(\tfrac{d}{dt})\, z}_{=0} \right\}$$

$$= \left\{ T^{-1}(\tfrac{d}{dt}) \underbrace{T(\tfrac{d}{dt})\, z}_{=y} \;\middle|\; S^{-1}(\tfrac{d}{dt})\, S(\tfrac{d}{dt})\, P(\tfrac{d}{dt})\, T(\tfrac{d}{dt})\, z = 0 \right\}$$

$$= \left\{ T^{-1}(\tfrac{d}{dt})\, y \;\middle|\; P(\tfrac{d}{dt})\, y = 0 \right\}$$

$$= T^{-1}(\tfrac{d}{dt}) \left\{ y \;\middle|\; P(\tfrac{d}{dt})\, y = 0 \right\}$$

$$= T^{-1}(\tfrac{d}{dt})\, \mathfrak{B}(P)$$

and thus also

$$\mathfrak{B}(PT) = T^{-1}(\tfrac{d}{dt})\, \mathfrak{B}(P) \qquad \Big| \to T(\tfrac{d}{dt})$$

$$T(\tfrac{d}{dt})\, \mathfrak{B}(PT) = \mathfrak{B}(P).$$

2.) $$\mathrm{image}_{\mathcal{C}_\infty}(T^{-1} U S) = \left\{ T^{-1}(\tfrac{d}{dt})\, U(\tfrac{d}{dt})\, S(\tfrac{d}{dt})\, \alpha \;\middle|\; \alpha \in \mathcal{C}_\infty^m \right\}$$

$$= T^{-1}(\tfrac{d}{dt}) \left\{ U(\tfrac{d}{dt}) \underbrace{S(\tfrac{d}{dt})\, \alpha}_{=:\beta} \;\middle|\; S^{-1}(\tfrac{d}{dt}) \underbrace{S(\tfrac{d}{dt})\, \alpha}_{=\beta} \in \mathcal{C}_\infty^m \right\}$$

$$= T^{-1}(\tfrac{d}{dt}) \left\{ U(\tfrac{d}{dt})\, \beta \;\middle|\; \beta = S(\tfrac{d}{dt})\, \alpha = S(\tfrac{d}{dt})\, \mathcal{C}_\infty^m = \mathcal{C}_\infty^m \right\}$$

Task 6:

"$\subseteq$" Let $z \in [T_1 \;\, T_2](\tfrac{d}{dt}) \begin{bmatrix} \mathfrak{U} \\ \mathfrak{B} \end{bmatrix}$, i.e.,

$\exists \; \alpha \in \mathfrak{U}, \; \beta \in \mathfrak{B}$ s.t.

$$z = T_1(\tfrac{d}{dt})\alpha + T_2(\tfrac{d}{dt})\beta$$

$$\in T_1(\tfrac{d}{dt})\mathfrak{U} + T_2(\tfrac{d}{dt})\mathfrak{B}$$

"$\supseteq$" Let $z \in T_1(\tfrac{d}{dt})\mathfrak{U} + T_2(\tfrac{d}{dt})\mathfrak{B}$, i.e.

$$z = T_1(\tfrac{d}{dt})\alpha + T_2(\tfrac{d}{dt})\beta \quad \text{with}$$

$$\alpha \in \mathfrak{U}, \; \beta \in \mathfrak{B}$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \begin{bmatrix} \mathfrak{U} \\ \mathfrak{B} \end{bmatrix}$$

$$\Rightarrow z = [T_1, T_2](\tfrac{d}{dt}) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in [T_1, T_2](\tfrac{d}{dt}) \begin{bmatrix} \mathfrak{U} \\ \mathfrak{B} \end{bmatrix}$$

To show that

$$T_1(\tfrac{d}{dt})\mathfrak{U} \cap T_2(\tfrac{d}{dt})\mathfrak{B} = \{0\}$$

Let z be with

$$z = T_1(\tfrac{d}{dt})\alpha = T_2(\tfrac{d}{dt})\beta,$$

where $\alpha \in \mathfrak{U}, \beta \in \mathfrak{B}$.

Let $\tilde{T} \in \mathbb{C}[\lambda]^{q,q}$ be the inverse of $[T_1, T_2]$

Then $0 = \tilde{z} - z = T_1\left(\frac{d}{dt}\right)\alpha - T_2\left(\frac{d}{dt}\right)\beta$

$$= \begin{bmatrix} T_1 , T_2 \end{bmatrix}\left(\frac{d}{dt}\right)\begin{bmatrix}\alpha \\ -\beta\end{bmatrix}.$$

Premultiplying by $\tilde{T}$ we find

$$0 = \tilde{T}\left(\frac{d}{dt}\right)\cdot 0 = \left(\tilde{T}\begin{bmatrix} T_1 , T_2 \end{bmatrix}\right)\left(\frac{d}{dt}\right)\begin{bmatrix}\alpha \\ -\beta\end{bmatrix}$$

$$= \begin{bmatrix}\alpha \\ -\beta\end{bmatrix} \Rightarrow \alpha = 0, \beta = 0.$$

<u>Task 7:</u>

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -\lambda L & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1/R & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ R & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -R & & 1 \\ & \lambda L & -1 \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} & \vdots & \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1/R & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & -\lambda L & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & \\ & 1 & 1 & 1/R \\ & \lambda L & & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & \\ & 1 & 1 & \frac{1}{R} \\ & & -\lambda L & \leftarrow \end{bmatrix} -\lambda \frac{L}{R} - 1$$

$$\Rightarrow P = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\lambda L & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 1 & \frac{1}{R} & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & \\ & 1 & 1 & \frac{1}{R} \\ & & -\lambda L & -\lambda\frac{L}{R} - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{R} & 0 \\ -\lambda L & -\lambda\frac{L}{R} & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{R} \\ 0 & 0 & -\lambda L & -\lambda\frac{L}{R} - 1 \end{bmatrix}$$

Task 7:

1.) We can apply the elementary unimodular transformations as follows:

$$\overset{(a)}{\circlearrowleft}\begin{bmatrix} 1 & 1 & 1 & \\ -R & & & 1 \\ & \lambda & & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & \\ & R & R & 1 \\ & \lambda & & -1 \end{bmatrix} \xrightarrow{(b)} \overset{(c)}{\circlearrowleft}\begin{bmatrix} 1 & & & \\ & R & R & 1 \\ & \lambda & & -1 \end{bmatrix} \to \begin{bmatrix} 1 & & & \\ & R & R & 1 \\ & \lambda+R & R & 0 \end{bmatrix} \quad (d)$$

$$\to \begin{bmatrix} 1 & & & \\ & 0 & 0 & 1 \\ & \lambda+R & R & 0 \end{bmatrix} \xrightarrow{(e)} \begin{bmatrix} 1 & & & \\ & 0 & 0 & 1 \\ & 0 & R & \end{bmatrix} \xrightarrow{(f)} \begin{bmatrix} 1 & & & \\ & 0 & 0 & 1 \\ & & 1 & \end{bmatrix} \quad (g)$$

$$\to \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix}.$$

In abstract notation we have to apply from the left

$$S(\lambda) := \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}_{(c)} \begin{bmatrix} 1 & & \\ R & 1 & \\ & & 1 \end{bmatrix}_{(a)} = \begin{bmatrix} 1 & 0 & 0 \\ R & 1 & 0 \\ R & 1 & 1 \end{bmatrix}$$

and from the right

$$T(\lambda) = \underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(b)} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -R & -R & 1 \end{bmatrix}}_{(d)} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\frac{\lambda+R}{R} & \frac{1}{R} & \\ & & & 1 \end{bmatrix}}_{(e \,\&\, f)} \begin{bmatrix} 1 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{bmatrix}$$

$$= \dots = \begin{bmatrix} 1 & 0 & -\frac{1}{R} & \lambda\frac{L}{R} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{R} & -\lambda\frac{L}{R}-1 \\ 0 & 1 & -1 & \lambda L \end{bmatrix}$$

Indeed, it holds

$$P = \begin{bmatrix} 1 & 0 & 0 \\ R & 1 & 0 \\ R & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{R} & \lambda\frac{L}{R} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{R} & -\lambda\frac{L}{R}-1 \\ 0 & 1 & -1 & \lambda L \end{bmatrix}}_{=T}$$

2.)

Lemma 1.10 $\Rightarrow$

$$U(\lambda) = \begin{bmatrix} \lambda\frac{L}{R} \\ 1 \\ -\lambda\frac{L}{R}-1 \\ \lambda L \end{bmatrix}$$ is the sought kernel spanning matrix.

3.) With the remarks after the Smith form we see that ~~(since $d_1 = \dots = d_3 = 1$)~~ (since $d_1 = \dots = d_3 = 1$ are nonzero constants) that

$$\mathcal{L}(P) = \mathrm{image}_{\mathcal{C}^\infty}\left(U\left(\tfrac{d}{dt}\right)\right)$$

4.) The Smith form is one possible echelon form.