Series 2

Task 1:

The MacMillan form of a full row rank rational matrix $R \in \mathbb{C}(\lambda)^{p,q}$ is given by

$$R = S\,[D, 0]\,T \quad \text{with}$$

$D = \operatorname{diag}\left(\frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_p}{\beta_p}\right)$ where $\beta_1 = \dots = \beta_p = 1$ are constants if R is polynomial.

Defining

$$(*) \qquad Q := T^{-1} \begin{bmatrix} D^{-1} \\ 0 \end{bmatrix} S^{-1} \quad \text{where}$$

$D^{-1} = \operatorname{diag}\left(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_p}{\alpha_p}\right)$ we find that

$$RQ = S\,[D, 0]\,T\,T^{-1} \begin{bmatrix} D^{-1} \\ 0 \end{bmatrix} S^{-1}$$

$$= S\,[D, 0] \begin{bmatrix} D^{-1} \\ 0 \end{bmatrix} S^{-1} = S S^{-1} = I$$

and the MacMillan form of Q is given by $(*)$. Thus

$$\mathcal{P}(R) = \mathcal{Z}(Q), \quad \mathcal{P}(Q) = \mathcal{Z}(R)$$

and Q is polynomial if and only if all $\alpha_1, \dots, \alpha_p$ are constant, i.e., $\mathcal{Z}(R) = \emptyset$.

Task 3: With the Smith form

$$P = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T =: S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$= S \begin{bmatrix} D T_1 \\ 0 \end{bmatrix}$$

we have

$$\underbrace{S^{-1}}_{=:U} P = \begin{bmatrix} D T_1 \\ 0 \end{bmatrix} =: \begin{bmatrix} \tilde{P} \\ 0 \end{bmatrix}$$

where $\tilde{P}$ has full row rank since T_1 has and D is invertible.

With this we conclude that

$$\mathfrak{B}(P) = \{ z \in \mathcal{C}_\infty^q \mid P(\tfrac{d}{dt}) z = 0 \}$$

$$= \{ z \in \mathcal{C}_\infty^q \mid U(\tfrac{d}{dt}) P(\tfrac{d}{dt}) z = U(\tfrac{d}{dt}) 0 = 0 \}$$

$$= \{ z \in \mathcal{C}_\infty^q \mid \begin{bmatrix} \tilde{P}(\tfrac{d}{dt}) z \\ 0 \end{bmatrix} = 0 \}$$

$$= \{ z \in \mathcal{C}_\infty^q \mid \tilde{P}(\tfrac{d}{dt}) z = 0 \} = \mathfrak{B}(\tilde{P}).$$

Task 4:

"$\Rightarrow$" Let $z \in \mathfrak{B}(P)$ with $z(t) = 0$, $t \leq 0$

Choose $z_2 \equiv 0 \in \mathfrak{B}(P)$.

$\Rightarrow z(t) = z_2(t)$, $t \leq 0$ $\Rightarrow z = z_2 = 0$

"$\Leftarrow$" Let $z_1, z_2 \in \mathfrak{B}(P)$ with $z_1(t) = z_2(t)$ for $t \leq 0$. Set $z := z_1 - z_2$ so that

$$z(t) = z_1(t) - z_2(t) = 0 \quad \forall t \leq 0$$

$$\Rightarrow 0 = z = z_1 - z_2 \quad \Rightarrow \quad z_1 = z_2$$

Task 5:

For $P \in \mathbb{C}[\lambda]^{p,q}$ the following are equivalent:

a) P is ~~left prime~~ right prime.

b) P has the Smith form
$$S \begin{bmatrix} I \\ 0 \end{bmatrix} T.$$

c) $\mathcal{Z}(P) = \emptyset \wedge \operatorname{rank}_{\mathbb{C}(\lambda)}(P) = q$

d) $p \geq q$ and there exists a matrix $\tilde{P} \in \mathbb{C}[\lambda]^{(p,(p-q)}$ such that $[P, \tilde{P}]$ is unimodular.

e) There exists a polynomial left inverse, i.e., a matrix $S \in \mathbb{C}[\lambda]^{q,p}$ with
$$SP = I$$
(Any such S is then left prime)

f) If $P = P_1 U$ for some $U \in \mathbb{C}[\lambda]^{q,q}$, $P_1 \in \mathbb{C}[\lambda]^{p,q}$ then U has to be unimodular.

Task 6c

1.) If U, V are kernel / cokernel matrices then $[U, V]$ is invertible.

$\Rightarrow$ U has full column rank

$\Rightarrow$ V has full column rank

2.) By Lemma 1.10 there exist kernel /
3.) cokernel matrices such that $[U, V]$ is unimodular. By Task 5 d) this implies that

U (and also V) are right prime.

Task 7:

$\mathcal{L} := \{ z_{\in \mathcal{C}_\infty^q} \mid \exists l_{\in \mathcal{C}_\infty^r} \text{ with } R(\frac{d}{dt}) z = M(\frac{d}{dt}) l \}$

linear:

Let $z_1, z_2 \in \mathcal{L}$, $\alpha \in \mathbb{C}$. ~~Then~~

$\Rightarrow \exists l_1, l_2$ with $R(\frac{d}{dt}) z_1 = M(\frac{d}{dt}) l_1$

$R(\frac{d}{dt}) z_2 = M(\frac{d}{dt}) l_2$

$$\Rightarrow R(\tfrac{d}{dt})(\alpha z_1 + z_2) = \alpha R(\tfrac{d}{dt}) z_1 + R(\tfrac{d}{dt}) z_2$$

$$= \alpha M(\tfrac{d}{dt}) l_1 + M(\tfrac{d}{dt}) l_2$$

$$= M(\tfrac{d}{dt}) \underbrace{(\alpha l_1 + l_2)}_{=: l}$$

$\Rightarrow \alpha z_1 + z_2 \in \mathcal{L}$

time invariant:

Let $z \in \mathcal{B}$ and $\ell \in \mathcal{C}_\infty^{\,}$ with

$$R(\tfrac{d}{dt}) z(t) = M(\tfrac{d}{dt}) \ell(t) \qquad \forall t \in \mathbb{R}$$

Let $\tau \in \mathbb{R}$ $\Rightarrow$ $R(\tfrac{d}{dt}) z(t+\tau) = M(\tfrac{d}{dt}) \underbrace{\ell(t+\tau)}_{=: \tilde{\ell}(t)}$

$$\Rightarrow \quad z(\cdot + \tau) \in \mathcal{B}.$$

Task 8:

2.) $\Rightarrow$ 3.) By definition left primeness means ~~we have~~

$$\operatorname{rank}_{\mathbb{C}} P(\lambda_0) = p \qquad \forall \lambda_0 \in \mathbb{C}.$$

However, right primeness in this case (square matrix) gives the same definition.

This shows that 1.) $\Leftrightarrow$ 2.) $\Leftrightarrow$ 3.)

2.) $\Leftrightarrow$ 4.) Theorem 1.13

2.) $\Leftrightarrow$ 7.) Theorem 1.13

4.) $\Leftrightarrow$ 5.) Lemma 1.5

5.) $\Leftrightarrow$ 6.) a scalar is unimodular if and only if it is a nonzero constant. Because if it was a higher order polynomial then the inverse would be a rational function

Task 9:

If z vanishes in I so do all its derivatives:

$$z^{(i)}(t) = 0 \qquad \forall t \in I, i \in \mathbb{N}_0.$$

~~$\Rightarrow P(\frac{d}{dt}) z =$~~ Thus, if we write P in the form $P(\lambda) = \lambda^k P_k + \ldots + \lambda P_1 + P_0$

we have that

$$P(\tfrac{d}{dt}) z(t) = P_k z^{(k)}(t) + \ldots + P_1 z^{(1)}(t) + P_0 z(t)$$
$$= 0 \qquad \forall t \in I.$$

and thus the claim.

Task 10:

1.) The system given by

$$0 = P(\tfrac{d}{dt}) z = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \frac{d}{dt}L & 0 & -1 \end{bmatrix} \begin{bmatrix} I_R \\ I_L \\ I \\ V \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -R \\ 0 \end{bmatrix} I_R + \begin{bmatrix} 1 \\ 0 \\ \frac{d}{dt}L \end{bmatrix} I_L + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} I + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} V$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ -R & 0 \\ 0 & \frac{d}{dt}L \end{bmatrix}}_{=: M(\frac{d}{dt})} \underbrace{\begin{bmatrix} I_R \\ I_L \end{bmatrix}}_{=: \ell, \text{ latent}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}}_{=: R(\frac{d}{dt})} \underbrace{\begin{bmatrix} I \\ V \end{bmatrix}}_{=: \tilde{z} \text{ manifest}}$$

is equivalent to the latent variable description ~~by~~

$$R(\tfrac{d}{dt})\tilde{z} = M(\tfrac{d}{dt})\,l \;, \text{ i.e.,}$$

$$R(\lambda) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad M(\lambda) = \begin{bmatrix} 1 & 1 \\ -R & 0 \\ 0 & \lambda L \end{bmatrix}.$$

As in the proof of Theorem 1.16 we compute a unimodular $U \in \mathbb{C}[\lambda]^{3,3}$ such that $UM = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$, where M_1 has full row rank, by the following elementary unimodular transformations

$$\begin{bmatrix} 1 & 1 \\ -R & 0 \\ 0 & \lambda L \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ -R & 0 \\ -\lambda L & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ -R & 0 \\ 0 & 0 \end{bmatrix}.$$

In abstract notation we apply from the left

$$U = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -\frac{\lambda L}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -\lambda L & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda L & -\lambda\frac{L}{R} & 1 \end{bmatrix}.$$

$$\Rightarrow \quad UR = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & -1 \\ \hline \lambda L & \lambda\frac{L}{R}+1 \end{array}\right].$$

Setting (similar to the proof of Theorem 1.16)

$$M_1(\lambda) := \begin{bmatrix} 1 & 1 \\ -R & 0 \end{bmatrix}, \quad R_1(\lambda) := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{R}(\lambda) = \left[\lambda L, \lambda\tfrac{L}{R}+1\right]$$

we have that

$$\mathcal{L} := \left\{ \tilde{z} \in \ell_\infty^2 \;\middle|\; R\left(\tfrac{d}{dt}\right)\tilde{z} = M\left(\tfrac{d}{dt}\right)\ell \right\}$$

$$= \left\{ \tilde{z} \in \ell_\infty^2 \;\middle|\; \begin{bmatrix} R_1\left(\frac{d}{dt}\right) \\ \tilde{R}\left(\frac{d}{dt}\right) \end{bmatrix} \tilde{z} = \begin{bmatrix} M_1\left(\frac{d}{dt}\right) \\ 0 \end{bmatrix} \ell \right\}$$

$$\overset{\text{Lemma 1.17 a)}}{=} \mathcal{L}(\tilde{R})$$

$$= \left\{ \begin{bmatrix} I \\ V \end{bmatrix} \in \ell_\infty^2 \;\middle|\; 0 = \tilde{R}\left(\tfrac{d}{dt}\right) \begin{bmatrix} I \\ V \end{bmatrix} \right.$$

$$\left. = L \cdot \dot{I} + \frac{L}{R}\dot{V} + V \right\}$$

$$= \left\{ \begin{bmatrix} I \\ V \end{bmatrix} \in \ell_\infty^2 \;\middle|\; -L \cdot \dot{I} = \frac{L}{R}\dot{V} + V \right\}$$

2.) In Series 1 the Smith form of P was computed as

$$P = S\,[I, 0]\,T.$$

Using Theorem 1.13 we conclude that P is prime.

$\tilde{P}$ is prime since both entries are coprime, i.e., $p_1(\lambda) = \lambda L$ and $p_2(\lambda) = \lambda \frac{L}{R} + 1$ in $\tilde{P} = [p_1, p_2]$ have no common zeros.

Thus $\operatorname{rank} \tilde{P}(\lambda_0) = 1 \quad \forall \lambda_0 \in \mathbb{C}$.

$\overset{\text{Def.}}{\Longrightarrow}$ $\tilde{P}$ is left prime.

3.) $\mathcal{L}(P)$ is not autonomous, since P can not have full column rank.

With the additional capacitor ~~resistor~~ the system is given by the matrix

$$\hat{P}(\lambda) := \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & \\ \hline -R & & & 1 \\ \hline & \lambda L & & -1 \\ \hline & & -1 & \lambda C \end{array}\right].$$

Computing ~~the "echelon form"~~ over $\underline{\mathbb{C}(\lambda)}$

$$\begin{bmatrix} 1 & 1 & 1 & \\ -R & & & 1 \\ & \lambda L & & -1 \\ & & -1 & \lambda C \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & \\ & R & R & 1 \\ & \lambda L & & -1 \\ & & -1 & \lambda C \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & \\ & R & R & 1 \\ & & -\lambda L & -1-\lambda\frac{L}{R} \\ & & -1 & \lambda C \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & -\lambda L & -1-\lambda\frac{L}{R} \\ 0 & 0 & 0 & \lambda L - \frac{1}{\lambda L}\left(-1-\lambda\frac{L}{R}\right) \end{bmatrix}$$

$$\lambda L - \frac{1}{\lambda L}\left(-1-\lambda\frac{L}{R}\right) = \lambda L + \frac{1}{\lambda L} + \frac{1}{R}$$

we see that ~~t~~ there exists an invertible (over $\mathbb{C}(\lambda)$) $S \in \mathbb{C}(\lambda)^{4,4}$ such that

$$S\hat{P} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & -\lambda L & -1-\lambda\frac{L}{R} \\ 0 & 0 & 0 & \lambda L + \frac{1}{\lambda L} + \frac{1}{R} \end{bmatrix}.$$

Since all diagonal elements of $S\hat{P}$ are nonzero rational functions $\hat{P}$ has full (row and column) rank (over $\mathbb{C}(\lambda)$) 4.

$\Rightarrow \mathcal{L}(\hat{P})$ is autonomous.

Task 11:

$$P = S \underbrace{\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}}_{\in \mathbb{C}(z)^{p,q}} T$$

$$= \underbrace{S \underbrace{\begin{bmatrix} D & \\ & I \end{bmatrix}}_{\in \mathbb{C}(z)^{p,p}}}_{=: \tilde{S} \in \mathbb{C}(z)^{p,p}} \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{\in \mathbb{C}(z)^{p,q}} T$$

$$= \tilde{S} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T \quad \text{, where}$$

$\tilde{S}$ and T are invertible (over $\mathbb{C}(z)$).

Task 12:

1.) By Task 3 there exists a unimodular U_a such that $U_a P = \begin{bmatrix} P_1 \\ 0 \end{bmatrix}$ where

P_1 has full row rank. Partition

$U_a Q =: \begin{bmatrix} Q_1 \\ \tilde{Q}_2 \end{bmatrix}$ accordingly. Using

Task 3 again there exists $\tilde{U}_b$ ~~a~~ unimodular such that $\tilde{U}_b \tilde{Q}_2 = \begin{bmatrix} Q_2 \\ 0 \end{bmatrix}$ where

Q_2 has full row rank.

Setting

$$U := \begin{bmatrix} I & 0 \\ 0 & \tilde{U}_b \end{bmatrix} U_a$$ we have

$$U[P, Q] = \begin{bmatrix} I & \\ & \tilde{U}_b \end{bmatrix} \begin{bmatrix} P_1 & \tilde{Q}_1 \\ 0 & \tilde{Q}_2 \end{bmatrix}$$

$$= \begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix}.$$

2.) If R has full column rank then also P and Q have full column rank. Thus also

$$UP = \begin{bmatrix} P_1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad UQ = \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix}$$

have full column rank.

Since then P_1 has full column and full row rank it is invertible.

Thus we see that

$$\operatorname{rank}_{\mathbb{C}(z)} \begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix} = \operatorname{rank}_{\mathbb{C}(z)} \begin{bmatrix} P_1 & 0 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix} \overset{\text{full column rank}}{=} q$$

$\Rightarrow Q_2$ has full column rank.

$\Rightarrow Q_2$ is invertible.