Homework, Series 3:

Task 1:

Block $\mathcal{L}_\varepsilon$: Size $\varepsilon \times (\varepsilon+1)$:

$$\mathcal{L}_\varepsilon = \begin{bmatrix} \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \lambda & -1 & \\ & & & \lambda & -1 \end{bmatrix} \xrightarrow{a1} \begin{bmatrix} \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \lambda & -1 & \\ & & & 0 & -1 \end{bmatrix} \rightarrow \dots \xrightarrow{a\varepsilon} \begin{bmatrix} 0 & -1 & & & \\ & \ddots & \ddots & & \\ & & 0 & -1 & \\ & & & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{b} \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ & & & 1 & 0 \end{bmatrix}$$

Thus the right transformation is

$$T := \underbrace{\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & \lambda & 1 \end{bmatrix}}_{a1} \cdot \dots \cdot \underbrace{\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{a\varepsilon} \cdot \underbrace{\begin{bmatrix} & & & 1 \\ -1 & & & 0 \\ & \ddots & & \vdots \\ & & -1 & 0 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} 1 & & & 0 \\ \lambda & \ddots & & \\ \vdots & \ddots & \ddots & \\ \lambda^\varepsilon & \dots & \lambda & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ -1 & & & \\ & \ddots & & \\ & & -1 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \lambda \\ \lambda & \ddots & & \vdots \\ \vdots & & 1 & \lambda^\varepsilon \end{bmatrix}$$

which means that the kernel is spanned by

$$U(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{\varepsilon} \end{bmatrix}$$ which is right prime (Theorem 1.13).

The behavior is then

$$\mathfrak{B}(\mathcal{L}_\varepsilon) = \left\{ \begin{bmatrix} z_0 \\ \vdots \\ z_\varepsilon \end{bmatrix} \in \mathcal{C}_\infty^{\varepsilon+1} \;\middle|\; 0 = \mathcal{L}_\varepsilon\left(\tfrac{d}{dt}\right) \begin{bmatrix} z_0 \\ \vdots \\ z_\varepsilon \end{bmatrix} \right.$$

$$= \begin{bmatrix} \frac{d}{dt} & -1 & & \\ & \ddots & \ddots & \\ & & \frac{d}{dt} & -1 \end{bmatrix} \begin{bmatrix} z_0 \\ \vdots \\ z_\varepsilon \end{bmatrix}$$

$\dot{z}_0 = z_1$

$\dot{z}_1 = z_2 \Rightarrow z_2 = \dot{z}_1 = \ddot{z}_0$

$\Rightarrow \dots \Rightarrow z_\varepsilon = z_0^{(\varepsilon)}$

$$\left. = \begin{bmatrix} \dot{z}_0 - z_1 \\ \vdots \\ \dot{z}_{\varepsilon-1} - z_\varepsilon \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} z_0 \\ z_0^{(1)} \\ \vdots \\ z_0^{(\varepsilon)} \end{bmatrix} \;\middle|\; z_0 \in \mathcal{C}_\infty^1 \right\}$$

$$= \operatorname{image}_{\mathcal{C}_\infty} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{\varepsilon} \end{bmatrix} = \operatorname{image}_{\mathcal{C}_\infty} U\left(\tfrac{d}{dt}\right).$$

Block $\mathcal{J}_\rho$: Size $\rho \times \rho$:

As in Example 1 on the handout "First order systems (Kronecker canonical form)" one shows that with

$$\mathcal{J}_\rho(\lambda) = \begin{bmatrix} \lambda-\lambda_j & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \lambda-\lambda_j \end{bmatrix}$$

$$S := \begin{bmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ (\lambda-\lambda_j) & \ddots & & \\ \vdots & \ddots & \ddots & \\ (\lambda-\lambda_j)^{\rho-1} & \cdots & (\lambda-\lambda_j) & 1 \end{bmatrix}$$

$$T := \begin{bmatrix} & & & 1 \\ 1 & & & (\lambda-\lambda_j) \\ & \ddots & & \vdots \\ & & 1 & (\lambda-\lambda_j)^{\rho-1} \end{bmatrix}$$

we have

$$S \mathcal{J}_\rho T = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & (\lambda-\lambda_j)^\rho \end{bmatrix}.$$

Thus $\mathcal{J}_\rho$ has full column rank which implies that the corresponding kernel spanning matrices has ~~full~~ zero columns.

The behavior is

$$\mathcal{L}(J_\rho) = \Big\{ z \in \mathcal{C}^\rho_\infty \;\Big|\; 0 = J_\rho\big(\tfrac{d}{dt}\big) z = \big(\tfrac{d}{dt} I - D\big) z = \dot{z} - Dz \Big\}$$

$$D := \begin{bmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{bmatrix}$$

$$= \Big\{ z \in \mathcal{C}^\rho_\infty \;\Big|\; \dot{z} = Dz \Big\}$$

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$$= \Big\{ z \in \mathcal{C}^\rho_\infty \;\Big|\; z(t) = e^{D \cdot t} \cdot z_0 \,,\; z_0 \in \mathbb{C}^\rho \Big\}$$

Block N_σ: Size $\sigma \times \sigma$

$$N_\sigma = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \xrightarrow{a1} \begin{bmatrix} 1 & 0 & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & \ddots & \lambda \\ & & & & 1 \end{bmatrix} \to \cdots \xrightarrow{a\sigma} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

Thus the right transformation is

$$T := \begin{bmatrix} 1 & -\lambda & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}^{a1} \cdots\cdots \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & -\lambda \\ 0 & & & 1 \end{bmatrix}^{a\sigma}$$

$$= \begin{bmatrix} 1 & (-\lambda) & \cdots & (-\lambda)^{\sigma-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & (-\lambda) \\ & & & 1 \end{bmatrix}.$$

The dimension of the kernel of N_σ is again zero which means that there is no real kernel spanning matrix (i.e., it has zero columns).

The behavior is

$$\mathfrak{B}(\mathcal{N}_\sigma) = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_\sigma \end{bmatrix} \;\middle|\; 0 = \mathcal{N}_\sigma(\tfrac{d}{dt}) \begin{bmatrix} z_1 \\ \vdots \\ z_\sigma \end{bmatrix} = \begin{bmatrix} z_1 - \dot z_2 \\ \vdots \\ z_{\sigma-1} - \dot z_\sigma \\ z_\sigma \end{bmatrix} \right\}$$

$\Rightarrow z_\sigma = 0$

$\Rightarrow \dot z_\sigma = 0$ $\qquad = \{0\}.$

$\Rightarrow 0 = z_{\sigma-1} - \dot z_\sigma = z_{\sigma-1}$

$\Rightarrow z_{\sigma-1} = 0 \Rightarrow \dots \Rightarrow z_1 = 0$

Block $\mathcal{M}_\eta$ is (almost) completely analogous to $\mathcal{N}_\sigma$.

4.) First compute the Kronecker canonical form and then use 1.) to compute the Smith form of each block separately. After further column and row permutations we arrive at the Smith form.

5.) With the notation $\Delta_r(\lambda) := \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^r \end{bmatrix} \in \mathbb{C}[\lambda]^{r+1,1}$

we saw in 1. that $\Delta_{\varepsilon_i}(\lambda)$ spans the kernel of $\mathcal{L}_{\varepsilon_i}(\lambda)$. This implies that

$$\begin{bmatrix} \mathcal{L}_{\varepsilon_1}(\lambda) & & \\ & \ddots & \\ & & \mathcal{L}_{\varepsilon_s}(\lambda) \end{bmatrix} \underbrace{\begin{bmatrix} \Delta_{\varepsilon_1}(\lambda) & & \\ & \ddots & \\ & & \Delta_{\varepsilon_s}(\lambda) \end{bmatrix}}_{=: U_{\mathcal{L}}(\lambda) \in \mathbb{C}[\lambda]^{\varepsilon, s}} = 0 .$$

One can show that $U_{\mathcal{L}}$ is a right prime polynomial kernel spanning matrix for $\mathcal{L} = \mathrm{diag}(\mathcal{L}_{\varepsilon_1}, \ldots, \mathcal{L}_{\varepsilon_s})$.

With $(\lambda F + G) = S \cdot \mathrm{diag}(\mathcal{L}, \mathcal{J}, \mathcal{N}, \mathcal{M}) \cdot T$

a right prime polynomial kernel spanning matrix for $(\lambda F + G)$ is then given by

$$T^{-1} \begin{bmatrix} U_{\mathcal{L}}(\lambda) \\ 0 \end{bmatrix},$$

since the other blocks have "no" kernel (over $\mathbb{C}(\lambda)$).

6.) See handout. (Lemma 5).

Task 2:

Blocks of type $\mathcal{J}$, $\mathcal{N}$, and $\mathcal{M}$ have full column rank (over $\mathbb{C}(\lambda)$). Thus they represent autonomous systems.

For blocks of type $\mathcal{L}$ (which have full row rank (over $\mathbb{C}(\lambda)$)) we can pick any single component as input, since by selecting ε-columns of

$$\mathcal{L}_\varepsilon(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \ddots \\ & & & \lambda & 1 \end{bmatrix}$$

results in

$$P(\lambda) = \begin{bmatrix} \lambda & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & \lambda & \ddots & \\ & & & & & \ddots & \ddots \\ & & & & & \lambda & 1 \end{bmatrix}$$

which is invertible (over $\mathbb{C}(\lambda)$).

Task 3:

Set $$P(\lambda) = \begin{bmatrix} \lambda-3 & -2 & -1 \\ 0 & \lambda-1 & 0 \end{bmatrix}$$

and perform

$$\xrightarrow{ⓐ} \begin{bmatrix} \lambda & 0 & -1 \\ & \lambda-1 & \end{bmatrix} \xrightarrow{ⓑ} \left[\begin{array}{cc|c} \lambda & -1 & \\ \hline & & \lambda-1 \end{array}\right]$$

which means the transformation

$$T = \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix}$$

gives

$$P(\lambda) \cdot T = \left[\begin{array}{cc|c} \lambda & -1 & \\ \hline & & \lambda-1 \end{array}\right]$$

Task 4:

Since $\ddot{x} = Ax + Bu$ is equivalent to the existence of a $v \in \mathcal{C}_\infty^n$ such that

$$\dot{\begin{bmatrix} x \\ v \end{bmatrix}} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u$$

one can give the following latent variable description:

$$\mathfrak{B}([\lambda^2 I - A, -B])$$

$$= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{C}_\infty^n \;\middle|\; \exists v \in \mathcal{C}_\infty^n \text{ s.t. } \begin{bmatrix} x \\ v \\ u \end{bmatrix} \in \mathfrak{B}\left(\begin{bmatrix} \lambda I & -I & 0 \\ -A & \lambda I & -B \end{bmatrix} \right) \right\}.$$

Task 5: Using Theorem 1.13 there exists a $\tilde{P}$ such that

$$\begin{bmatrix} P \\ \tilde{P} \end{bmatrix}$$ is unimodular. Thus also

$$\begin{bmatrix} U & \\ & I \end{bmatrix} \begin{bmatrix} P \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} UP \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \tilde{P} \end{bmatrix}$$ is unimodular.

Using Theorem 1.13 again this shows that P_1 and P_2 are left prime.

Task 6: 1.) Using Series 2, Task 8 we remember that a matrix is unimodular if and ~~a~~ only if it has full (row and column) rank and no zeros.

Using point 3. of Lemma 4 from the handout "First order systems / Kronecker canonical form" we see that U and its canonical linearization have the same zeros.

Furthermore, with point 1 of the same Lemma we see that (if $U(\lambda) = \lambda^k U_k + \ldots$) the rank of U and its canonical linearization $\lambda Q_1 + Q_0$ is ~~b~~ related by

$$\operatorname{rank}_{\mathbb{C}(\lambda)} (\lambda Q_1 + Q_0) = p(k-1) + \operatorname{rank}_{\mathbb{C}(\lambda)} U .$$

which means that U has full rank if

and only if the (pK)-by-(pK) matrix $\lambda Q_1 + Q_0$ has full rank.

2.) Same argument as in 1.); but use Theorem 1.13 (instead of Lemma 4 from the handout)

Task 7:

Partition the signal space as $\begin{array}{cl} \left[\begin{array}{c} y \\ \hline u \end{array}\right] & \begin{array}{l} l \text{ elements} \\ m \text{ elements} \end{array} \end{array}$ such that $P \in \mathbb{C}[\lambda]^{p,l}$, $Q \in \mathbb{C}[\lambda]^{p,m}$.

Then we have the equivalences

$$\left[\operatorname{rank}_{\mathbb{C}(\lambda)} P = l = \operatorname{rank}_{\mathbb{C}(\lambda)} R \right] \iff$$

$$\left[\operatorname{rank}_{\mathbb{C}(\lambda)} P = l = p \right] \iff$$

$$\left[P \in \mathbb{C}[\lambda]^{p,p} \text{ is invertible} \right]$$

Task 8:

1.) Since $\begin{bmatrix} 1 & 1 & 1 \\ -R & 0 & 0 \\ 0 & \lambda L & 0 \end{bmatrix}$ is invertible (over $\mathbb{C}(\lambda)$)

the partition $\left(\begin{bmatrix} 1 & 1 & 1 \\ -R & 0 & 0 \\ 0 & \lambda L & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$ is

input/output, i.e., we can choose V input.

Since $\begin{bmatrix} 1 & 1 & 0 \\ -R & 0 & 1 \\ 0 & \lambda L & -1 \end{bmatrix}$ is invertible (over $\mathbb{C}(\lambda)$)

the partition $\left(\begin{bmatrix} 1 & 1 & 0 \\ -R & 0 & 1 \\ 0 & \lambda L & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$ is also

input/output, i.e., we can choose I input.

In the same way one sees that both other variables can be chosen input.

2.) After a lot of trying one finds that

$$\begin{bmatrix} 1 & 1 & 1 & \\ -R & & & 1 \\ & \lambda L & & -1 \end{bmatrix} \xrightarrow{a} \begin{bmatrix} 1 & 1 & 1 & \\ & R & R & 1 \\ & \lambda L & & -1 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 & & & \\ & R & R & 1 \\ & \lambda L & & -1 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 1 & & & \\ & & R & \\ & \lambda & & -1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & \lambda & & -1 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} & \lambda & & -1 \\ & & 1 & \\ 1 & & & \end{bmatrix} \xrightarrow{f} \left[\begin{array}{cc|cc} \lambda & -1 & & \\ \hline & & \boxed{1} & \\ & & & 1 \end{array}\right]$$

which corresponds to the right transformation

$$T = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & \frac{1}{R} & -\frac{1}{R} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\qquad\qquad\qquad\qquad$ b $\qquad\qquad\qquad\qquad$ c & d $\qquad\qquad\qquad\qquad$ f

$$
= \dots = \begin{bmatrix} 0 & \frac{1}{R} & -\frac{1}{R} & 1 \\ \frac{1}{L} & 0 & 0 & 0 \\ -\frac{1}{L} & -\frac{1}{R} & \frac{1}{R} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

and to the left transformation

$$
S = \underbrace{\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}}_{(e)} \cdot \underbrace{\begin{bmatrix} 1 & & \\ R & 1 & \\ & & 1 \end{bmatrix}}_{(a)} = \begin{bmatrix} & & 1 \\ R & 1 & \\ 1 & & \end{bmatrix}
$$

to obtain that the Kronecker canonical form is

$$
S \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix} T = \left[\begin{array}{cc|c|c} \lambda & -1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right]
$$

Task 9:

1.) Let $AV = VJ$ be the Jordan form with
$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix} \text{ with}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, \dots, r.$$

Choosing $\lambda_0 \neq \lambda_i$ for all $i = 1, \dots, r$ we find that $(\lambda_0 I - J_i)$ is invertible for all $i = 1, \dots, r$. Thus with Lemma 1.9 we have

$$n \geq \operatorname{rank}_{\mathbb{C}(\lambda)} (\lambda I - A) = \operatorname{rank}_{\mathbb{C}(\lambda)} (V^{-1}(\lambda I - A)V)$$
$$= \operatorname{rank}_{\mathbb{C}(\lambda)} (\lambda I - J)$$
$$= \max_{\hat{\lambda} \in \mathbb{C}} \operatorname{rank} (\hat{\lambda} I - J) \geq \operatorname{rank} (\lambda_0 I - J)$$
$$= \sum_{i=1}^{r} \operatorname{rank} (\lambda_0 I - J_i) = n$$

which shows that $(\lambda I - A)$ is regular.

2.) If P_k is invertible then the "highest order coefficient" of the canonical linearization
$$\begin{bmatrix} I & & \\ & \ddots & \\ & & I \\ & & & P_k \end{bmatrix} \text{ is invertible.}$$

Using 1. this shows that the canonical linearization is invertible.
Using Lemma ~~3~~4, 1.) from the handout this implies that also P is invertible.

3.) If P_k has full row rank then there exists an invertible (over $\mathbb{C}$) matrix Π such that in

$$P_k \Pi = [\underbrace{\tilde{P}_k}_{p \text{ cols}}, \underbrace{\tilde{Q}_k}_{q-p \text{ cols}}]$$

the matrix $\tilde{P}_k$ is invertible. Set

$$P_i \Pi =: [\underbrace{\tilde{P}_i}_{p \text{ cols}}, \underbrace{\tilde{Q}_i}_{q-p \text{ cols}}]$$

and define $\tilde{P}(\lambda) := \sum_{i=0}^{k} \lambda^i \tilde{P}_i$, $\tilde{Q}(\lambda) := \sum_{i=0}^{k} \lambda^i \tilde{Q}_i$.

Using 2.) we see that $\tilde{P}$ is invertible (over $\mathbb{C}(\lambda)$). Thus

$$P(\lambda) \cdot \Pi = [\tilde{P}(\lambda), \tilde{Q}(\lambda)]$$

has full row rank (over $\mathbb{C}(\lambda)$) and thus also P itself.

4.) Counterexample

$$P(\lambda) = \left[\begin{array}{c|c} \lambda^2+1 & 0 \\ 0 & \lambda-1 \end{array}\right]$$

Task 10:

Use Theorem 1.19 & Theorem 1.21.

1.) With $R(\lambda) = [\lambda I - A, -B]$ the system is not autonomous, since R has fewer rows than columns:

$$\operatorname{rank}_{\mathbb{C}(\lambda)} R \leq n$$

Indeed, by Task 9 we have

$$\operatorname{rank}_{\mathbb{C}(\lambda)} R = n = \operatorname{rank}_{\mathbb{C}(\lambda)} (\lambda I - A)$$

which means that $(\lambda I - A, -B)$ is an input/output partition.

2.) Similar to 1.) $(\lambda^2 I - A, -B)$ is an input/output partition of

$$\mathcal{L}([\lambda^2 I - A, -B]).$$

3.) Since $(\lambda E - A)$ has full column rank the system $\mathcal{L}(\lambda E - A)$ is autonomous. Thus there are no free components and the exists no input/output partition.

4.) Under this assumption the system is not autonomous but an input/output partition is not easily specified.

5.) With $R(\lambda) = [M\lambda^2 + D\lambda + K,\ -B_1\lambda - B]$

the system is not autonomous (same argument as in 1.). Since $(\lambda^2 M + \lambda D + K)$ is invertible (Task 8) an input/output partition is $(\lambda^2 M + \lambda D + K,\ -B_1\lambda - B)$.

6.) With
$$R(\lambda) = \left[\begin{array}{c|c} \lambda^2 M - K & -B \\ \hline 0 & \lambda I - D \end{array}\right]$$

the system is autonomous, since R is invertible:

It is a block upper triangular matrix with invertible blocks on the diagonal.