Series 4: Homework

Task 1:

1.) We have

$$V^*(t_0, t_1) = \left(\int_{t_0}^{t_1} G(s) G^*(s)\, ds \right)^*$$

$$= \int_{t_0}^{t_1} (G(s) G^*(s))^* \, ds$$

$$= \int_{t_0}^{t_1} G(s) G^*(s)\, ds = V(t_0, t_1)$$

and for all $x_0 \in \mathbb{C}^n$ we have

$$x_0^* V(t_0, t_1) x_0 = x_0^* \left(\int_{t_0}^{t_1} G(s) G^*(s)\, ds \right) x_0$$

$$= \int_{t_0}^{t_1} x_0^* G(s) G^*(s) x_0 \, ds$$

$$= \int_{t_0}^{t_1} \underbrace{\| G^*(s) x_0 \|_2^2}_{\geq 0} \, ds \geq 0.$$

2.) "$\subseteq$": Let $x_0 \in \operatorname{kernel}(V(t_0, t_1))$

$\Rightarrow 0 = V(t_0, t_1) x_0$

$\Rightarrow 0 = x_0^* V(t_0, t_1) x_0 = \ldots = \int_{t_0}^{t_1} \| G^*(s) x_0 \|_2^2 \, ds$

$\Rightarrow G^*(t) x_0 = 0 \quad \forall t \in [t_0, t_1]$, since otherwise the integral would be positive

"$\supseteq$": Let $x_0 \in \mathbb{C}^n$ with $G^*(t)x_0 = 0$.

Then

$$V(t_0,t_1)\, x_0 = \int_{t_0}^{t_1} G(s) \underbrace{G^*(s) x_0}_{=0} ds = 0.$$

3.) "$\subseteq$": Let $x_0 \in \text{image } V(t_0,t_1)$.

$$\Rightarrow x_0 = \left(\int_{t_0}^{t_1} G(s)\, G^*(s)\, ds \right) \cdot \alpha$$

$$= \int_{t_0}^{t_1} G(s) \underbrace{G^*(s)\alpha}_{=:u} ds \quad \in \mathcal{R}.$$

"$\supseteq$": For the remark let $x_0 \in \text{kernel}\, V(t_0,t_1) \cap \mathcal{R}$

Then by 2. we have

$$G^*(s)\, x_0 = 0 \qquad \forall s \in [t_0,t_1]$$

and there exists a $u \in \mathcal{C}^m_\infty$ such that

$$x_0 = \int_{t_0}^{t_1} G(s) u(s)\, ds$$

$$\Rightarrow \|x_0\|_2^2 = x_0^* x_0 = x_0^* \int_{t_0}^{t_1} G(s) u(s)\, ds$$

$$= \int_{t_0}^{t_1} \underbrace{(G^*(s) x_0)^*}_{=0} u(s) ds = 0$$

$\Rightarrow$ kernel $V(t_0, t_1) \cap \mathcal{W} = \{0\}$.

The claim then follows from the following

Lemma:

Let $A \in \mathbb{C}^{n,n}$ and $\mathcal{W} \subset \mathbb{C}^n$ a linear subspace.
Assume that

i) image $A \subseteq \mathcal{W}$

ii) kernel $A \cap \mathcal{W} = \{0\}$.

Then image $A = \mathcal{W}$.

Proof:

~~Since~~ From ii) we deduce that

$$(\dim \operatorname{kernel}(A)) + (\dim \mathcal{W}) \leq n \qquad (1)$$

since otherwise the intersection could not be trivial.

Furthermore, in linear algebra it is shown that

$$(\dim \operatorname{kernel}(A)) + (\dim \operatorname{image} A) = n. \qquad (2)$$

Thus, we have

$$(\dim \operatorname{image} A) \overset{(2)}{=} n - (\dim \operatorname{kernel}(A))$$

$$\overset{(1)}{\geq} (\dim \operatorname{kernel}(A)) + (\dim \mathcal{W}) - (\dim \operatorname{ker...})$$

$= \dim \mathcal{W}$. Since i) implies

$(\dim \operatorname{image} A) \leq \dim \mathcal{W}$ we have $(\dim \operatorname{image} A) = \dim \mathcal{W}$ and thus (with i) again) image $A = \mathcal{W}$. $\blacksquare$

Task 2:

$$\left[x_0 \in C(t_1, t_0) \right] \iff$$

$$\left[\exists (x, \hat{u}) \in \mathcal{C}_\infty^{n+m} \text{ with } \dot{x} = A(t)x + B(t)\hat{u} \text{ and } x(t_0) = x_0 , x(t_1) = 0 \right] \overset{\text{Thm 1.24}}{\iff}$$

$$\Big[\exists \hat{u} \in \mathcal{C}_\infty^{m} \text{ with } 0 = x(t_1) = \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, s) B(s) \hat{u}(s)\, ds \Big]$$

Lemma 1.25 a)

$$\iff \Big[\Phi(t_1, t_0) x_0 = - \int_{t_0}^{t_1} \Phi(t_1, t_0)\, \Phi(t_0, s) B(s) u(s)\, ds = \Phi(t_1, t_0) \int_{t_0}^{t_1} \Phi(t_0, s) B(s) \underbrace{(-\hat{u}(s))}_{=: u(s)}\, ds \Big]$$

Lemma 1.25 b)

$$\iff \Big[x_0 = \int_{t_0}^{t_1} \Phi(t_0, s) B(s) u(s)\, ds \overset{\text{Task 1, c)}}{=} \operatorname{image} V(t_1, t_0) \Big]$$

Task 3:

$$\left[x_0 \in C(t_1, t_0) \right] \iff \left[\exists (\hat{x}, \hat{u}) \in \mathcal{C}_\infty^{n+m} \text{ with } \dot{\hat{x}} = A\hat{x} + B\hat{u} , \ \hat{x}(t_0) = x_0 , \ \hat{x}(t_1) = 0 \right] \iff$$

$$\left[\exists (x, u) \in \mathcal{C}_\infty^{n+m} \text{ with } x = Ax + Bu , \ x(0) = \hat{x}(0 + t_0) = x_0 , \ x(t_1 - t_0) = \hat{x}(t_1 - t_0 + t_0) = \hat{x}(t_1) = 0 \right]$$

where for the last equivalence the "both-way def." $x(t) :=: \hat{x}(t + t_0)$ was used.

Task 4:

"$\Rightarrow$": Let $x_0, x_1 \in \mathbb{C}^n$, $\tau > 0$.

Then we have

$$\mathcal{C}(\tau) \underset{\text{Thm 2.4, i)}}{=} \text{image } V(\tau) \underset{\text{Thm 2.6}}{=} \text{image } K(A,B) = \mathbb{C}^n$$

and also

$$\mathcal{R}(\tau) \underset{\text{Thm 2.4ii)}}{=} e^{\tau A} \text{ image } V(\tau) \underset{\text{Thm 2.6}}{=} e^{\tau A} \cdot \text{image } K(A,B)$$

$$= \underbrace{(e^{\tau A})}_{\substack{\text{invertible} \\ \text{Lem. 1.25}}} \cdot \mathbb{C}^n = \mathbb{C}^n$$

$\Rightarrow$ $x_0 \in \mathcal{C}(\tau)$ ⓐ, $x_1 \in \mathcal{R}(\tau)$ ⓑ.

$\overset{ⓐ}{\Rightarrow}$ $\exists (\hat{x}_0, \hat{u}_0) \in \mathcal{C}_\infty^{n+m}$ with $\dot{\hat{x}}_0 = A\hat{x}_0 + B\hat{u}_0$

and $\hat{x}_0(0) = x_0$, $\hat{x}_0(\tau) = 0$

$\overset{ⓑ}{\Rightarrow}$ $\exists (\hat{x}_1, \hat{u}_0) \in \mathcal{C}_\infty^{n+m}$ with $\dot{\hat{x}}_1 = A\hat{x}_1 + B\hat{u}_1$

and $\hat{x}_1(\tau) = x_1$, $\hat{x}_1(0) = 0$.

Setting $x := (\hat{x}_0 + \hat{x}_1)$, $u := (\hat{u}_0 + \hat{u}_1)$

shows that $\dot{x} = \dot{\hat{x}}_0 + \dot{\hat{x}}_1 = A\hat{x}_0 + B\hat{u}_0 + A\hat{x}_1 + B\hat{u}_1$

$= Ax + Bu$

and $\quad x(0) = \hat{x}_1(0) + \hat{x}_0(0) = x_0$

$$x(\tau) = \hat{x}_1(\tau) + \hat{x}_0(\tau) = x_1 .$$

"$\Leftarrow$" Let $x_0 \in \mathbb{C}^n$ be arbitrary. In this case there especially exists a $(u,x) \in \mathcal{C}_\infty^{n+m}$ such that

$$\dot{x} = Ax + Bu \quad , \quad x(0) = x_0 \quad , x(1) = 0.$$

$$\Rightarrow \quad x_0 \in C(1) \quad \Rightarrow \quad C(1) = \mathbb{C}^n$$

$$\Rightarrow \quad \mathbb{C}^n = C(1) = \operatorname{image} V(1) = \operatorname{image} K(A,B).$$

$\Rightarrow$ $K(A,B)$ has full row rank.

<u>Task 5:</u> We have

$$\operatorname{image} K(A,B) = \left\{ B\alpha_0 + AB\alpha_1 + \dots + A^{n-1}B\alpha_{n-1} \,\middle|\, \alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}^m \right\}$$

$$= \left\{ B\underbrace{\alpha_0}_{=:\beta_0} + ((-1)A)B\underbrace{((-1)\alpha_1)}_{=:\beta_1} + \dots + ((-1)A)^{n-1}B\underbrace{((-1)^{n-1}\alpha_{n-1})}_{=:\beta_{n-1}} \,\middle|\, \alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}^m \right\}$$

$$= \left\{ B\beta_0 + (-A)B\beta_1 + \dots + (-A)^{n-1}B\beta_{n-1} \,\middle|\, \beta_0, \dots, \beta_{n-1} \in \mathbb{C}^m \right\}$$

$= \operatorname{image} K(-A,B)$ and similar for the others. Thus we have $\operatorname{rank} K(A,B) = \operatorname{rank} K(-A,B) = \dots$ and the claim is shown.

Task 6:

The first linear system can be written as

$$\underbrace{\begin{bmatrix} \frac{d}{dt}I - A \\ -C \end{bmatrix}}_{=:M(\frac{d}{dt})} \underbrace{x}_{=:\ell} = \underbrace{\begin{bmatrix} B & 0 \\ D & -I \end{bmatrix}}_{=:R(\frac{d}{dt})} \underbrace{\begin{bmatrix} u \\ y \end{bmatrix}}_{=:z} .$$

Then a latent variable description is given by

$$\mathcal{L}_e := \{ z \mid \exists \ell \text{ s.t. } R(\tfrac{d}{dt}) z = M(\tfrac{d}{dt}) \ell \}.$$

Similar, for the second system one can set

$$\tilde{M}(\lambda) := \left[\begin{array}{c} \lambda I - VAV^{-1} \\ \hline -CV^{-1} \end{array}\right], \tilde{R}(\lambda) := \left[\begin{array}{c|c} VB & 0 \\ \hline D & -I \end{array}\right]$$

so that one can write a latent variable description in the form

$$\tilde{\mathcal{L}}_e := \{ \tilde{z} \mid \exists \tilde{\ell} \text{ s.t. } \tilde{R}(\tfrac{d}{dt}) \tilde{z} = \tilde{M}(\tfrac{d}{dt}) \tilde{\ell} \}$$

Then we have

$$\mathcal{L} = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \;\middle|\; \exists x : \begin{bmatrix} \frac{d}{dt}I - A \\ -C \end{bmatrix} x = \begin{bmatrix} B & 0 \\ D & -I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \;\middle|\; \exists x : \begin{bmatrix} V & \\ & I \end{bmatrix} \begin{bmatrix} \frac{d}{dt}I - A \\ -C \end{bmatrix} V^{-1} \underbrace{Vx}_{=z} \right.$$

$$\left. = \begin{bmatrix} V & \\ & I \end{bmatrix} \begin{bmatrix} B & 0 \\ D & -I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \;\middle|\; \exists z : \left[\begin{array}{c} \frac{d}{dt}VV^{-1} - VAV^{-1} \\ \hline -CV^{-1} \end{array} \right] z = \left[\begin{array}{c|c} VB & 0 \\ \hline D & -I \end{array} \right] \begin{bmatrix} u \\ y \end{bmatrix} \right\}$$

$$= \tilde{\mathcal{L}} .$$

Task 7:

ⓐ The system is

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}}_{=:A} \begin{bmatrix} q \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{=:B} f$$

and thus the controllability matrix is

$$K(A,B) = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{d}{m^2} \end{bmatrix}.$$

For $m > 0$ $K(A,B)$ has full row rank. Thus the system is controllable.

ⓑ The system is

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} q_1 \\ q_2 \\ v_1 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ \frac{1}{m_2} \end{bmatrix}}_{=:B} f$$

and thus the controllability matrix is

$$K(A,B) = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & -\frac{k_1}{m_1^2} \\ 0 & \frac{1}{m_2} & 0 & -\frac{k_2}{m_2^2} \\ \frac{1}{m_1} & 0 & -\frac{k_1}{m_1^2} & 0 \\ \frac{1}{m_2} & 0 & -\frac{k_2}{m_2^2} & 0 \end{bmatrix}.$$

It has full (row) rank if and only if

$$\begin{bmatrix} \frac{1}{m_1} & -\frac{k_1}{m_1^2} \\ \frac{1}{m_2} & -\frac{k_2}{m_2^2} \end{bmatrix} \quad \text{has full rank.}$$

Since

$$\det\left(\begin{bmatrix} \frac{1}{m_1} & -\frac{k_1}{m_1^2} \\ \frac{1}{m_2} & -\frac{k_2}{m_2^2} \end{bmatrix}\right) = -\frac{k_2}{m_1 m_2^2} + \frac{k_1}{m_2 m_1^2} = 0$$

$$\Leftrightarrow \quad -m_1 k_2 + m_2 k_1 = 0$$

$$\Leftrightarrow \quad m_1 k_2 = m_2 k_1$$

the system is controllable if and only if

$$m_1 k_2 \neq m_2 k_1 .$$