Series 7:

Task 1:

1.) $\Rightarrow$ 2.)

$$\mathfrak{B}(R) = \mathfrak{B}\left(\begin{bmatrix}P\\C\end{bmatrix}\right) = \left\{ z \in \mathcal{C}_\infty^q \;\middle|\; \begin{matrix} P(\frac{d}{dt})z = 0 \\ C(\frac{d}{dt})z = 0 \end{matrix} \right\}$$

$$= \mathfrak{B}(P) \cap \mathfrak{B}(C) \subseteq \mathfrak{B}(P).$$

2.) $\Rightarrow$ 1.) With $C := R$ we have

$$\mathfrak{B}\left(\begin{bmatrix}P\\C\end{bmatrix}\right) = \ldots = \mathfrak{B}(P) \cap \mathfrak{B}(C)$$

$\mathfrak{B}(R) \subseteq \mathfrak{B}(P)$

$$= \mathfrak{B}(P) \cap \mathfrak{B}(R) = \mathfrak{B}(R)$$

Task 2:

$p = q = 1$:

Assume to the contrary that $P \in \mathbb{C}[\lambda]$ is not zero. Then there exists a $\lambda_0 \in \mathbb{C}$ such that $P(\lambda_0) \neq 0$.

Setting $\hat{z}(t) := e^{\lambda_0 t} \in \mathcal{C}_\infty^1$ shows that

$$P(\tfrac{d}{dt})\hat{z}(t) = \underbrace{P(\lambda_0)}_{\neq 0} \cdot \underbrace{e^{\lambda_0 t}}_{\neq 0} \neq 0 \quad \forall t \in \mathbb{R}.$$

$\Rightarrow \hat{z} \notin \mathfrak{B}(P)$ which contradicts to assumption $\mathfrak{B}(P) = \mathcal{C}_\infty^1$.

$p \geq 1$, $q = 1$: For $P = \begin{bmatrix}P_1\\ \vdots \\ P_p\end{bmatrix} \in \mathbb{C}[\lambda]^{p,1}$ we have

$$\mathscr{C}_\infty^1 = \mathscr{L}(P) = \bigcap_{i=1}^{p} \mathscr{L}(P_i)$$

$$\Rightarrow \mathscr{L}(P_i) = \mathscr{C}_\infty^1 \quad \forall i \in \{1,\dots,p\}$$

$$\overset{p=1=q}{\Rightarrow} P_i = 0 \quad \forall i \in \{1,\dots,p\} \Rightarrow P = 0.$$

$p \geq 1, q \geq 1$:

With the notation $P = [P_1, \dots, P_q] \in \mathbb{C}[\lambda]^{p,q}$, $P_1, \dots, P_q \in \mathbb{C}[\lambda]^{p,1}$ we have for all $i = 1, \dots, q$ that

$$0 \underset{\substack{\uparrow \\ \text{assumption}}}{=} P(\tfrac{d}{dt})(e_i \alpha(t)) = P_i(\tfrac{d}{dt}) \alpha(t) \quad \forall \alpha \in \mathscr{C}_\infty^1.$$

$$\Rightarrow \mathscr{L}(P_i) = \mathscr{C}_\infty^1 \overset{p\geq 1, q=1}{\Rightarrow} P_i = 0 \Rightarrow P = 0.$$

Task 3:

$p = r = 1$: Let $D, P \in \mathbb{C}[\lambda]$.

If $P = 0$ one can choose $M = 0$. Thus we can w.l.o.g assume that $P \neq 0$.

We show that $\mathcal{Z}(D) \overset{①}{\subseteq} \mathcal{Z}(P)$ by contradiction.

Because if ① would not be the case there would be a $\lambda_0 \in \mathbb{C}$ with

$$D(\lambda_0) = 0 \quad \text{but} \quad P(\lambda_0) \overset{P \neq 0}{\neq} 0.$$

$$\Rightarrow e^{\lambda_0 t} \in \mathscr{L}(D), \quad e^{\lambda_0 t} \notin \mathscr{L}(P),$$

which contradicts the assumption.

Also we have $D \neq 0$, since otherwise the assumption would imply $\mathcal{L}(P) = \{0\}$

$\overset{\text{Task 2}}{\Rightarrow} \quad P = 0 \quad ⚡.$

Thus $\mathcal{Z}(D) \subseteq \mathcal{Z}(P)$ means that there exist $c_1, c_2, \lambda_1, \dots, \lambda_m \in \mathbb{C}$ such that

$$D(\lambda) = c_1 (\lambda - \lambda_1) \cdots (\lambda - \lambda_d)$$

$$P(\lambda) = c_2 (\lambda - \lambda_1) \cdots (\lambda - \lambda_d)(\lambda - \lambda_{d+1}) \cdots (\lambda - \lambda_m)$$

with $d \leq m$, $c_1, c_2 \neq 0$.

$$\Rightarrow \quad P(\lambda) = D(\lambda) \cdot \underbrace{\frac{c_2}{c_1} (\lambda - \lambda_{d+1}) \cdots (\lambda - \lambda_m)}_{=: M}$$

$p \geq 1, r = 1$:

With $P = \begin{bmatrix} P_1 \\ \vdots \\ P_p \end{bmatrix} \in \mathbb{C}[\lambda]^{p,1}$ the assumption reads

$$\mathcal{L}(D) \subseteq \mathcal{L}(P) = \bigcap_{i=1}^{p} \mathcal{L}(P_i)$$

$$\Rightarrow \quad \mathcal{L}(D) \subseteq \mathcal{L}(P_i)$$

$$\overset{(p=1,\, r=1)}{\Rightarrow} \quad \exists M_i \in \mathbb{C}[\lambda] \text{ with } M_i D = P_i$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} M_1 \\ \vdots \\ M_p \end{bmatrix}}_{=: M} D = \underbrace{\begin{bmatrix} P_1 \\ \vdots \\ P_p \end{bmatrix}}_{= P}$$

$p \geq 1, r \geq 1$:

With the notation $P = [P_1, \ldots, P_r] \in \mathbb{C}[\lambda]^{p,r}$, $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{bmatrix} \in \mathbb{C}[\lambda]^{r,r}$ the assumption implies that

$$d_i(\tfrac{d}{dt})\alpha(t) = 0 \quad \Rightarrow \quad P_i(\tfrac{d}{dt})\alpha(t) = 0 \quad \forall i = 1, \ldots, r.$$

$\overset{p\geq 1, r=1}{\Rightarrow}$ $\exists\, M_i \in \mathbb{C}[\lambda]^{p,1}$ such that

$$M_i\, d_i = P_i$$

$$\Rightarrow \underbrace{[M_1, \ldots, M_r]}_{=:M} \underbrace{\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{bmatrix}}_{=D} = \underbrace{[P_1, \ldots, P_r]}_{=P}.$$

Task 4:

"$\Leftarrow$" "$\subseteq$" Let $z \in \mathfrak{L}(MD)$.

$$\Rightarrow M(\tfrac{d}{dt})\underbrace{D(\tfrac{d}{dt})z}_{=:y} = 0 \quad \Rightarrow \quad y \in \mathfrak{L}(M)$$

$$\overset{\text{Assumption}}{\Rightarrow} y = 0 \quad \Rightarrow \quad D(\tfrac{d}{dt})z = 0 \Rightarrow z \in \mathfrak{L}(D)$$

"$\supseteq$" Let $z \in \mathfrak{L}(D)$.

$$\Rightarrow D(\tfrac{d}{dt})z = 0 \quad \Rightarrow \quad M(\tfrac{d}{dt})D(\tfrac{d}{dt})z = 0$$

$$\Rightarrow z \in \mathfrak{L}(MD).$$

"$\Rightarrow$" "$\subseteq$" Let $z \in \mathfrak{L}(M)$

$$\Rightarrow M(\tfrac{d}{dt}) z = 0$$

$$\overset{\text{Lemma 1.17 a)}}{\Rightarrow} \exists \alpha \in \mathcal{C}_\infty^{r} \text{ such that } z = D(\tfrac{d}{dt})\alpha,$$

since D has full row rank.

$$\Rightarrow M(\tfrac{d}{dt}) D(\tfrac{d}{dt}) \alpha = 0 \quad \Rightarrow \alpha \in \mathfrak{L}(MD)$$

$$\Rightarrow \alpha \in \mathfrak{L}(D) \Rightarrow 0 = D(\tfrac{d}{dt})\alpha = z$$

$$\Rightarrow z = 0$$

"$\supseteq$" trivial.

Task 5: Choose $P(\lambda) = (\lambda - 1)$, $R(\lambda) = 1$.
Then $\{ce^t \mid c \in \mathbb{C}\} = \mathfrak{L}(P) \neq \mathfrak{L}(R) = \{0\}$.
Let $C \in \mathbb{C}[\lambda]^{s,1}$ be with

$$\mathfrak{L}(\begin{bmatrix} P \\ C \end{bmatrix}) = \mathfrak{L}(R) = \{0\}.$$

Then $C \neq 0$ since otherwise

$$\mathfrak{L}(\begin{bmatrix} P \\ C \end{bmatrix}) = \mathfrak{L}(\begin{bmatrix} P \\ 0 \end{bmatrix}) = \mathfrak{L}(P) \neq \mathfrak{L}(R) = \{0\}. \text{ ↯}$$

$$\Rightarrow \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} P}_{=1} + \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} C}_{=1} > \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} P \\ C \end{bmatrix}}_{=1}.$$

Task 6: Set $\Lambda := \mathcal{Z}(P)$ in ~~Lemma~~ Theorem 3.4 to obtain a $\boxed{\text{regular}}$ controller $C \in \mathbb{C}[\lambda]^{s,q}$ such that:

i) $\mathcal{Z}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = \Lambda = \mathcal{Z}(P) \subseteq \mathbb{C}_-$

Thm 2.19 P is stabilizable

ii) $\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$ is autonomous

iii) $\mathcal{Z}(C) = \Lambda / \mathcal{Z}(P) = \emptyset$.

By Lemma 3.3 and i), ii) this shows that $\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$ is stable. Thus C is a $\boxed{\text{stabilizable}}$ controller.

Since C is regular, C also has full row rank. Since by iii) $\mathcal{Z}(C) = \emptyset$ this shows that C is left prime by Theorem 1.13.

Task 7:

If C_1 and C_2 are regular then with the notation $c := c_1 + c_2$, $q := q_1 + q_2$

$$C := \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} \in \mathbb{C}[z]^{c,q}, \quad P := \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}$$

we have

$$c = c_1 + c_2 \overset{\text{Assumption}}{=} \operatorname{rank}_{\mathbb{C}(z)} C_1 + \operatorname{rank}_{\mathbb{C}(z)} C_2$$
$$= \operatorname{rank}_{\mathbb{C}(z)} C$$

and

$$\operatorname{rank}_{\mathbb{C}(z)} \begin{bmatrix} P \\ C \end{bmatrix} - \operatorname{rank}_{\mathbb{C}(z)} P = \operatorname{rank}_{\mathbb{C}(z)} \left[\begin{array}{c|c} P_1 & \\ & P_2 \\ \hline C_1 & \\ & C_2 \end{array}\right] - \operatorname{rank}_{\mathbb{C}(z)} \left[\begin{array}{c|c} P_1 & \\ & P_2 \end{array}\right]$$

$$= \operatorname{rank}_{\mathbb{C}(z)} \left[\begin{array}{c|c} P_1 & \\ C_1 & \\ \hline & P_2 \\ & C_2 \end{array}\right] - \operatorname{rank}_{\mathbb{C}(z)} P_1 - \operatorname{rank}_{\mathbb{C}(z)} P_2$$

$$= \underbrace{\operatorname{rank}_{\mathbb{C}(z)} \begin{bmatrix} P_1 \\ C_1 \end{bmatrix} - \operatorname{rank}_{\mathbb{C}(z)} P_1}_{} + \underbrace{\operatorname{rank}_{\mathbb{C}(z)} \begin{bmatrix} P_2 \\ C_2 \end{bmatrix} - \operatorname{rank}_{\mathbb{C}(z)} P_2}_{}$$

$$\overset{\text{Assumption 6}}{=} c_1 + c_2 = c$$, i.e., C is regular.

If C_1 and C_2 are stabilizing controller this means (by Lemma 3.3) that

$\mathfrak{B}\left(\begin{bmatrix}P_i\\C_i\end{bmatrix}\right)$, $i=1,2$ are autonomous and

$$\mathcal{Z}\left(\begin{bmatrix}P_i\\C_i\end{bmatrix}\right) \subseteq \mathbb{C}_- \qquad i=1,2.$$

This implies that $\begin{bmatrix}P_i\\C_i\end{bmatrix}$, $i=1,2$ have full column rank (Theorem 1.19). Thus also

$$\operatorname{rank}_{\mathbb{C}(\lambda)}\begin{bmatrix}P\\C\end{bmatrix} = \operatorname{rank}_{\mathbb{C}(\lambda)}\left[\begin{array}{c|c}P_1 & \\ & P_2\\ \hline C_1 & \\ & C_2\end{array}\right] = \operatorname{rank}_{\mathbb{C}(\lambda)}\left[\begin{array}{c|c}P_1 & \\ C_1 & \\ \hline & P_2\\ & C_2\end{array}\right],$$

~~rank~~ i.e., $\begin{bmatrix}P\\C\end{bmatrix}$ has full column rank and thus $\mathfrak{B}\left(\begin{bmatrix}P\\C\end{bmatrix}\right)$ is autonomous. Furthermore, we have

$$\mathcal{Z}\left(\begin{bmatrix}P\\C\end{bmatrix}\right) = \mathcal{Z}\left(\left[\begin{array}{c|c}P_1 & \\ & P_2\\ \hline C_1 & \\ & C_2\end{array}\right]\right) = \mathcal{Z}\left(\left[\begin{array}{c|c}P_1 & \\ C_1 & \\ \hline & P_2\\ & C_2\end{array}\right]\right)$$

$$= \mathcal{Z}\left(\begin{bmatrix}P_1\\C_1\end{bmatrix}\right) \cup \mathcal{Z}\left(\begin{bmatrix}P_2\\C_2\end{bmatrix}\right) \subseteq \mathbb{C}_-$$

which shows (again by Lemma 3.3) that $\mathfrak{B}\left(\begin{bmatrix}P\\C\end{bmatrix}\right)$ is stable.

Task 8:

Assume to the contrary that $C \neq 0$ would be regular. Then we would have

$$1 \le \operatorname{rank}_{\mathbb{C}(\lambda)} C = \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} P \\ C \end{bmatrix} - \operatorname{rank}_{\mathbb{C}(\lambda)} P$$

rank is smaller then ~~ade~~ number of columns

$$\le q - \operatorname{rank}_{\mathbb{C}(\lambda)} P$$

P has full column rank

$$= q - q = 0. \quad \lightning$$

Task 9:

By Lemma 3.1 there exists a unimodular N such that $P_1 = N P_2$.

Thus
$$\mathfrak{B}(P_1) = \mathfrak{B}(N P_2) = \mathfrak{B}(P_2).$$

Task 10:

1.) The controlled system is the kernel of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \\ \hline 0 & 0 & -1 & \lambda C \end{bmatrix}.$$

Applying elementary unimodular transformations to this matrix we find

$$
\begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \\ 0 & 0 & -1 & \lambda C \end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 & \\ & R & R & 1 \\ & \lambda L & & -1 \\ & & -1 & \lambda C \end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & -\lambda L & -1-\lambda\frac{L}{R} \\ & & -1 & \lambda C \end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 & 0 \\ & R & R & 1 \\ & & -1 & \lambda C \\ & & -\lambda L & -1-\lambda\frac{L}{R} \end{bmatrix}
$$

$$
\sim
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & -1 & \lambda L \\ 0 & 0 & 0 & -1-\lambda\frac{L}{R}-\lambda^2 LC \end{bmatrix}
. \qquad (*)
$$

This shows that

$$
1 = \mathrm{rank}_{\mathbb{C}(\lambda)} \underbrace{[0, 0, -1, \lambda C]}_{=: \tilde{C}(\lambda)} = \cancel{\mathrm{rank}_{\mathbb{C}(\lambda)}}
$$

and

$$
\mathrm{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} \lambda F + G \\ \tilde{C}(\lambda) \end{bmatrix} - \mathrm{rank}_{\mathbb{C}(\lambda)} (\lambda F + G) \overset{(*)}{=} 4 - 3 = 1
$$

$\Rightarrow$ $\tilde{C}$ is regular.

Also $\mathfrak{L}\left(\begin{bmatrix} \lambda F + G \\ \tilde{C}(\lambda) \end{bmatrix}\right)$ is autonomous by $(*)$

and the zeros are

$$
Z\left(\begin{bmatrix} \lambda I + G \\ \tilde{C}(\lambda) \end{bmatrix}\right) = Z\left(-1 - \lambda\frac{L}{R} - \lambda^2 LC\right)
$$

$$
= Z(R + \lambda L + \lambda^2 RLC)
$$

$$= \left\{ - \frac{L \pm \sqrt{L^2 - 4LCR^2}}{2CLR} \right\} \subseteq \mathbb{C}_-$$

$\Rightarrow$ stabilizing controller.

2.) Resistor: Begining as in 1.) we obtain

$$\begin{bmatrix} 1 & 1 & 1 & \\ -R & & & 1 \\ & \lambda L & & -1 \\ & & \tilde{R} & -1 \end{bmatrix} \sim \dots \sim \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 0 \\ & R & R & 1 \\ & & \tilde{R} & -1 \\ & & -\lambda L & -1-\lambda\frac{L}{R} \end{array}\right]$$

$$\sim \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & \tilde{R} & -1 \\ 0 & 0 & 0 & -1-\lambda\frac{L}{R}-\lambda\frac{L}{\tilde{R}} \end{array}\right] \Rightarrow \text{regular}$$

$$\Rightarrow \mathcal{Z}(\dots) = \mathcal{Z}\left(1 + \lambda L\left(\frac{1}{R} + \frac{1}{\tilde{R}}\right)\right)$$

$$= \left\{ -\frac{1}{L}\left(\frac{R\tilde{R}}{R+\tilde{R}}\right) \right\}. \Rightarrow \text{stabilizing}$$

Inductor: Begining as in 1. we obtain

$$\begin{bmatrix} 1 & 1 & 1 & \\ -R & & & 1 \\ & \lambda L & & -1 \\ & & \lambda \tilde{L} & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & \lambda\tilde{L} & -1 \\ 0 & 0 & -\lambda L & -1-\lambda\frac{L}{R} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & \lambda\tilde{L} & -1 \\ 0 & 0 & 0 & -1-\lambda\frac{L}{R}-\frac{L}{\tilde{L}} \end{bmatrix}$$

$\Rightarrow$ regular

$$\Rightarrow \mathcal{Z}(\ldots) = \mathcal{Z}(\lambda \tilde{L}) \cup \mathcal{Z}\left(-1 - \frac{L}{\tilde{L}} - \lambda \frac{L}{R}\right)$$

$$= \{0\} \cup \left\{-R\left(\frac{1}{L} + \frac{1}{\tilde{L}}\right)\right\}$$

$\Rightarrow$ stabilizing.

3.) As in 1.) and 2.) we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \\ 0 & 0 & c_I(\lambda) & c_V(\lambda) \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & 0 & c_I(\lambda) & c_V(\lambda) \\ 0 & 0 & -\lambda L & -1-\lambda\frac{L}{R} \end{bmatrix} =: M(\lambda)$$

Since here we $\cancel{\neq}$ want

$$\underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} M}_{\geq 3} - \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} P}_{=3} \neq \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} [0, 0, c_I, c_V]}_{=1}$$

we have to make sure that $\operatorname{rank}_{\mathbb{C}(\lambda)} M = 3$. This is then equivalent to

$$\operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} c_I(\lambda) & c_V(\lambda) \\ -\lambda L & -1-\lambda\frac{L}{R} \end{bmatrix} = 1.$$

Thus, one can choose

$$c_I(\lambda) := (-\lambda L)\, p(\lambda)$$

$$c_V(\lambda) := \left(-1-\lambda\frac{L}{R}\right) p(\lambda)$$

with an arbitrary $p \in \mathbb{C}[\lambda]$ $\cancel{, p \neq 0}$.

Task 11:

1.) First observe that

$$(*) \quad P(\lambda) \sim \left[\begin{array}{c|c} \lambda^2 m + \lambda(c+d) + k & -1 \\ \hline 0 & -1 \end{array}\right].$$

Then we see that

$$\text{rank}_{\mathbb{C}(\lambda)} [\lambda c \ , -1] = 1 \quad \text{and}$$

$$\text{rank}_{\mathbb{C}(\lambda)} \left[\begin{array}{c|c} \lambda^2 m + \lambda d + k & -1 \\ \lambda c & -1 \end{array}\right] - \text{rank}_{\mathbb{C}(\lambda)} [\lambda^2 m + \lambda d + k]$$

$$= 2 - 1 = 1 \quad \Rightarrow \text{regular (independent of } c\text{).}$$

Since $(*)$ implies that P has full column rank, $\mathfrak{B}(P)$ is also autonomous (independent of c).

2.) Using the formula from Series 5 we see that

$$\mathfrak{Z}(P) = \mathfrak{Z}(\lambda^2 m + \lambda(c+d) + k)$$

$$= \left\{ -\frac{d+c}{2m} \pm \sqrt{\left(\frac{d+c}{2m}\right)^2 - \frac{k}{m}} \right\}.$$

Thus there are two cases

1. $\left(\frac{d+c}{2m}\right)^2 < \frac{k}{m}$ (underdamped)

$$\Rightarrow \max_{\lambda \in \mathfrak{Z}(P)} \text{Re}(\lambda) = -\frac{d+c}{2m} =: f_1(c)$$

$$\Rightarrow f_1'(c) = -\frac{1}{2m} < 0 \qquad \text{(decreasing)}$$

$\underline{2.}$ $\left(\frac{d+c}{2m}\right)^2 > \frac{k}{m}$ (overdamped)

$$\Rightarrow \max_{\lambda \in \mathcal{Z}(P)} \mathrm{Re}(\lambda) = -\frac{d+c}{2m} + \sqrt{\left(\frac{d+c}{2m}\right)^2 - \frac{k}{m}} =: f_2(c)$$

$$\Rightarrow f_2'(c) = -\frac{1}{2m} + \frac{1}{2}\,\frac{2\left(\frac{d+c}{2m}\right)\cdot\frac{1}{2m}}{\sqrt{\left(\frac{d+c}{2m}\right)^2 - \frac{k}{m}}}$$

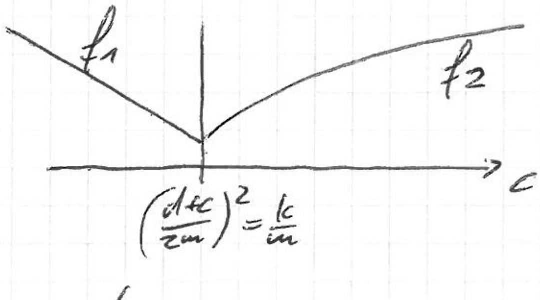
$$= \frac{1}{2m}\left(-1 + \frac{\sqrt{\left(\frac{d+c}{2m}\right)^2}}{\sqrt{\left(\frac{d+c}{2m}\right)^2 - \frac{k}{m}}}\right)$$

Expression is still real since $\left(\frac{d+c}{2m}\right)^2 > \frac{k}{m}$

$$> \frac{1}{2m}\left(-1 + \frac{\sqrt{\left(\frac{d+c}{2m}\right)^2 - \frac{k}{m}}}{\sqrt{\left(\frac{d+c}{2m}\right)^2 - \frac{k}{m}}}\right) = \frac{1}{2m}(-1+1) = 0$$

(increasing).

Thus,
We are in the following situation



and the optimum is assumed for

$$\left[\left(\frac{d+c}{2m}\right)^2 = \frac{k}{m}\right] \iff \dots \iff \boxed{c = 2\sqrt{km} - d}$$