Series 8:

Task 1:

For the special case $\varepsilon = 3$ we see that via elementary unimodular transformations we have

$$\begin{bmatrix} \lambda & -1 & & \\ & \lambda & -1 & \\ & & \lambda & -1 \\ c_0 & c_1 & c_2 & c_3 \end{bmatrix} \sim \begin{bmatrix} \lambda & -1 & & \\ & \lambda & -1 & \\ & & 0 & -1 \\ c_0 & c_1 & c_2+\lambda c_3 & c_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} \lambda & -1 & & \\ & 0 & -1 & \\ & & 0 & -1 \\ c_0 & c_1+\lambda c_2+\lambda^2 c_3 & c_2+\lambda c_3 & c_3 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & & \\ & 0 & -1 & \\ & & 0 & -1 \\ p(\lambda) & c_1+\lambda c_2+\lambda^2 & c_2+\lambda c_3 & c_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -1 & & \\ & 0 & -1 & \\ & & 0 & -1 \\ p(\lambda) & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & \\ & p(\lambda) \end{bmatrix}.$$

In the same way we obtain for the general case that there exist unimodular S, T such that

$$S(\lambda) \begin{bmatrix} \mathcal{L}_\varepsilon(\lambda) \\ C \end{bmatrix} T(\lambda) = \begin{bmatrix} I_\varepsilon & \\ & p(\lambda) \end{bmatrix}$$

$\Rightarrow \mathcal{Z}\left(\begin{bmatrix} \mathcal{L}_\varepsilon(\lambda) \\ C \end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix} I_\varepsilon & \\ & p(\lambda) \end{bmatrix}\right) = \mathcal{Z}(p)$ and also

$$\operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} \mathcal{L}_\varepsilon(\lambda) & \\ C & \end{bmatrix} = \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} I_\varepsilon & \\ & p(\lambda) \end{bmatrix}$$

$$= \varepsilon + \operatorname{rank}_{\mathbb{C}(\lambda)} p(\lambda) = \operatorname{rank}_{\mathbb{C}(\lambda)} \mathcal{L}_\varepsilon(\lambda) + \operatorname{rank}_{\mathbb{C}(\lambda)} p(\lambda)$$

Task 2:

(a) We have

$$\underbrace{\begin{bmatrix} I & BC_2^{-1} \\ 0 & I \end{bmatrix}}_{=:S_1} \underbrace{\begin{bmatrix} \lambda I - A & -B \\ C_1 & C_2 \end{bmatrix}}_{=P(\lambda)} \underbrace{\begin{bmatrix} I & \\ -C_2^{-1}C_1 & I \end{bmatrix}}_{=:T_1}$$

$$= \begin{bmatrix} I & BC_2^{-1} \\ & I \end{bmatrix} \begin{bmatrix} \lambda I - A + BC_2^{-1}C_1 & -B \\ 0 & C_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda I - A + BC_2^{-1}C_1 & 0 \\ 0 & C_2 \end{bmatrix} =: \widetilde{P}(\lambda)$$

Since by Series 3, Task 9 matrix polynomials of the form $(\lambda I - \dots)$ are invertible and also C_2 is invertible we conclude that $\widetilde{P}(\lambda)$ and thus also

$$P(\lambda) = S_1^{-1}\, \widetilde{P}(\lambda)\, T_1^{-1} \quad \text{is invertible.}$$

(b) We have

$$m \underset{\substack{\uparrow \\ \text{rank smaller than number of rows}}}{\geq} \operatorname{rank} [\, C_1 \,,\, C_2 \,] \geq \operatorname{rank} C_2 = m$$

$\Rightarrow \quad m = \operatorname{rank} C_2$

$$\Rightarrow \quad \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} \left[\begin{array}{c|c} \lambda I - A & -B \\ \hline C_1 & C_2 \end{array}\right]}_{\overset{(a)}{=} n+m} - \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} [\lambda I - A, -B]}_{\overset{S.3,T.9}{=} n}$$

$$= n + m - \cancel{m}n = \operatorname{rank} C_2.$$

(c) Choose $A = [1]$, $B = [1]$, $C_1 = [1]$, $C_2 = [0]$

$$\Rightarrow \operatorname{rank}_{\mathbb{C}(\lambda)} P(\lambda) = \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} \lambda - 1 & -1 \\ 1 & 0 \end{bmatrix} = 2$$

$$\operatorname{rank}_{\mathbb{C}(\lambda)} [\lambda - 1, -1] = 1$$

$$\operatorname{rank} [1, 0] = 1$$

$\Rightarrow [1, 0]$ is regular controller for $\mathfrak{B}([\lambda-1, -1])$.

Task 3:

$$\mathfrak{B}\left(\left[\begin{array}{c|c} P & 0 \\ \hline -R & I \end{array}\right]\right) = \left\{ (z_1, z_2) \in \mathcal{C}_\infty^{q+r} \;\middle|\; \begin{array}{l} P(\tfrac{d}{dt}) z_1 = 0 \\ -R(\tfrac{d}{dt}) z_1 + z_2 = 0 \end{array} \right\}$$

$$= \left\{ (z_1, z_2) \mid z_1 \in \mathfrak{B}(P), \; z_2 = R(\tfrac{d}{dt}) z_1 \right\}$$

$$= \left\{ (z_1, R(\tfrac{d}{dt}) z_1) \mid z_1 \in \mathfrak{B}(P) \right\} = \begin{bmatrix} I \\ R(\tfrac{d}{dt}) \end{bmatrix} \mathfrak{B}(P).$$

Task 4: ~~Write~~ Use the notation $R := \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$.

(a) If R_{11} has full row rank over $\mathbb{C}(\lambda)$ there exists an $X \in \mathbb{C}(\lambda)^{q_1, p_1}$ such that

$$R_{11}(\lambda) X(\lambda) = I. \quad \text{(cf. Series 2, Task 1)}$$

$$\Rightarrow \operatorname{rank}_{\mathbb{C}(\lambda)} R = \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} R_{11} & R_{12} \\ & R_{22} \end{bmatrix} \begin{bmatrix} I & -XR_{12} \\ & I \end{bmatrix}$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix} = \operatorname{rank}_{\mathbb{C}(\lambda)} R_{11} + \operatorname{rank}_{\mathbb{C}(\lambda)} R_{22}$$

$$= p_1 + q_2$$

To show (1):

"$\subseteq$": Let $\lambda_0 \in \mathcal{Z}(R)$, with Lemma 1.9

$$\operatorname{rank} R(\lambda_0) < \operatorname{rank}_{\mathbb{C}(\lambda)} R = p_1 + q_2$$

Then $\lambda_0 \in \mathcal{Z}(R_{11})$ or, if this not the case, we have $\operatorname{rank} R_{11}(\lambda_0) = p_1$ and thus there exists a $\tilde{X} \in \mathbb{C}^{q_1, p_1}$ such that $R_{11}(\lambda_0) \tilde{X} = I$.

$$\Rightarrow \operatorname{rank} R(\lambda_0) = \operatorname{rank} \begin{bmatrix} R_{11}(\lambda_0) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda_0) \end{bmatrix} \begin{bmatrix} I & -\tilde{X} R_{12}(\lambda_0) \\ 0 & I \end{bmatrix}$$

$$= \operatorname{rank} \begin{bmatrix} R_{11}(\lambda_0) & 0 \\ 0 & R_{22}(\lambda_0) \end{bmatrix} = p_1 + \operatorname{rank} R_{22}(\lambda_0)$$

$$\Rightarrow \quad p_1 + q_2 > \operatorname{rank} R(\lambda_0) = p_1 + \operatorname{rank} R_{22}(\lambda_0)$$

$$\Rightarrow \quad q_2 > \operatorname{rank} R_{22}(\lambda_0) \quad \Rightarrow \quad \lambda_0 \in \mathcal{Z}(R_{22}).$$

"$\supseteq$": If $\lambda_0 \in \mathcal{Z}(R_{11}) / \mathcal{Z}(R_{22})$ then $\operatorname{rank} R_{22}(\lambda_0) = q_2$ and as before we get

$$\operatorname{rank} R(\lambda_0) = \operatorname{rank} R_{11}(\lambda_0) + q_2 < p_1 + q_2$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} R \overset{\text{Lem. 1.9.}}{\Rightarrow} \lambda_0 \in \mathcal{Z}(R)$$

If $\lambda_0 \in \mathcal{Z}(R_{22}) / \mathcal{Z}(R_{11})$ an analogous argument implies.

It remains the case when $\lambda_0 \in \mathcal{Z}(R_{11}) \cap \mathcal{Z}(R_{22})$, i.e., when $\operatorname{rank} R_{11}(\lambda_0) < p_1$, $\operatorname{rank} R_{22}(\lambda_0) < q_2$.

$$\Rightarrow \operatorname{rank} R(\lambda_0) \leq \operatorname{rank} [R_{11}(\lambda_0) \quad R_{12}(\lambda_0)] + \operatorname{rank} R_{22}(\lambda_0)$$

rank is smaller than number of rows

$$\leq p_1 + \operatorname{rank} R_{22}(\lambda_0) < p_1 + q_2$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} R.$$

$$\Rightarrow \lambda_0 \in \mathcal{Z}(R).$$

(b) Choose $R_{11}(\lambda) = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$, $R_{12}(\lambda) = \begin{bmatrix} 0 & 0 \\ \lambda-1 & 0 \end{bmatrix}$,

$R_{22}(\lambda) = [0, \lambda]$ $\Rightarrow$ $R(\lambda) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$.

Task 5:

Choose $C = [1, 0, \dots, 0]$.

Then (in the special case $\varepsilon = 3$; the general case is analogous) we see that by applying elementary unimodular transformations from the left that

$$\mathcal{L}\left(\begin{bmatrix} \lambda & -1 & & \\ & \lambda & -1 & \\ & & \lambda & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{L}\left(\begin{bmatrix} 0 & -1 & & \\ & \lambda & -1 & \\ & & \lambda & -1 \\ 1 & & & \end{bmatrix}\right) = \mathcal{L}\left(\begin{bmatrix} & -1 & & \\ & 0 & -1 & \\ & & \lambda & -1 \\ 1 & & & \end{bmatrix}\right)$$

$$= \mathcal{L}\left(\begin{bmatrix} & -1 & & \\ & & -1 & \\ & & & -1 \\ 1 & & & \end{bmatrix}\right) = \{0\}$$

$\overset{\text{S.5, T.5}}{\Rightarrow}$ $\begin{bmatrix} \lambda & -1 & & \\ & \lambda & -1 & \\ & & \lambda & -1 \\ 1 & & & \end{bmatrix}$ is right prime $\Rightarrow \mathcal{Z}\left(\begin{bmatrix} \lambda & -1 & & \\ & \lambda & -1 & \\ & & \lambda & -1 \\ 1 & & & \end{bmatrix}\right) = \emptyset$.

Task 6:

$$P(\lambda) = \left[\begin{array}{c|c|cc|c} 0 & \lambda I - A & 0 & 0 & -B \\ \hline I & -C & 0 & 0 & 0 \\ L & 0 & -L & \lambda I - A & -B \\ 0 & 0 & I & -C & 0 \end{array}\right]$$

Task 7:

1.) If $\mathcal{L}([\lambda I - A, -B])$ is stabilizable then $\mathcal{Z}([\lambda I - A, -B]) \subseteq \mathbb{C}_-$ and the Kronecker canonical form of

$$\lambda F + G = \lambda [I, 0] + [-A, -B]$$

only contains blocks of type $\mathcal{L}$ and $\mathcal{J}$, where all zeros of the $\mathcal{J}$ blocks are in the left half plane.

In the notation of Theorem 2, handout "Constant controllers..." this means that

$$n = p = \varepsilon + \rho \quad \text{and} \quad n+m = q = \varepsilon + \rho + s$$

$$\Rightarrow m = s$$

and thus there exists a regular, stabilizing controller

$$C = [C_1, C_2] \in \mathbb{C}^{s, n+m}$$

with

$$\operatorname{rank} \begin{bmatrix} F \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix} = n+m$$

$$\Rightarrow C_2 \text{ is invertible}$$

Setting $G := -C_2^{-1} C_1$ (as in Corollary 3.6) this implies that

$$\mathcal{L}\left(\begin{bmatrix} \lambda I - A & -B \\ C_1 & C_2 \end{bmatrix}\right) = \mathcal{L}\left(\begin{bmatrix} \lambda I - A & -B \\ -G & I \end{bmatrix}\right)$$

is stable.

$$\overset{\text{as in Corollary 3.6}}{\Longrightarrow} \quad \sigma(A + BG) \subseteq \mathbb{C}_- .$$

2.) The assumption implies that

$$\mathbb{C}_- \supseteq \mathcal{Z}\left(\begin{bmatrix} \lambda I - A \\ -C \end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix} \lambda I - A^T , -C^T \end{bmatrix}\right)$$

$$\Rightarrow \mathcal{L}\left(\begin{bmatrix} \lambda I - A^T , -C^T \end{bmatrix}\right) \text{ is stabilizable}$$

$$\overset{1.)}{\Longrightarrow} \exists \tilde{G} \text{ s.t. } \quad \sigma(A^T + C^T \tilde{G}) \subseteq \mathbb{C}_-$$

$$\Rightarrow \mathbb{C}_- \supseteq \sigma(A^T + C^T \tilde{G}) = \sigma(A + \underbrace{\tilde{G}^T}_{=:L} C) .$$

3.) Combine 1.) & 2.) and use Corollary 3.8.

<u>Task 8:</u> As in the proof of Theorem 2 from the handout we first see that for the block of type $\mathcal{L}(\lambda)$ we have

$$\mathcal{Z}\left(\begin{bmatrix} [\lambda \mid -1] \\ C_1 \end{bmatrix}\right) \underset{\substack{\uparrow \\ \text{choose } C_1 := [d \ -1]}}{=} \mathcal{Z}\left(\begin{bmatrix} \lambda & -1 \\ d & -1 \end{bmatrix}\right) = \cancel{\mathcal{Z}\left(\begin{bmatrix} 0 & -1 \\ \lambda - d & -1 \end{bmatrix}\right)}$$

$$= \mathcal{Z}\left(\begin{bmatrix} 0 & -1 \\ \lambda - d & -1 \end{bmatrix}\right)$$

$$= \{d\}$$

and thus we choose the controller

$$C := [C_1, 0] \cdot T$$

$$= [d, -1, 0, 0] \begin{bmatrix} 0 & L & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & R & R & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [0, L \cdot d, 0, -1].$$

Task 9:

1.) Let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq 0$, $\lambda \in \mathbb{C}$ be an eigenpair of $\begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}$. $\Rightarrow$ $\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -Kv_1 \end{bmatrix}$

$$\Rightarrow \lambda v_1 = v_2 \quad \Rightarrow -Kv_1 = \lambda v_2 = \lambda^2 v_1$$

$$\Rightarrow \lambda^2 \in \sigma(-K).$$

Since $-K$ is ~~pos~~ negative semi-definite this implies that

$$\lambda^2 \in \mathbb{R}_{\leq 0}$$

$$\Rightarrow \lambda \in i\mathbb{R} \text{ is purely imaginary.}$$

2.) In this case A has the form of 1.) with

$$K := \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \text{ and we}$$

only have to show that this K is positive semi-definite. This is the case if and only if

$$S^T \begin{bmatrix} k_1 + k_2 & -k_2 \\ k_2 & k_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{= S}$$

$$= \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} k_1 & -k_2 \\ & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & \\ & k_2 \end{bmatrix}$$ is positive semi-definite.

The latter, however is the case, since $k_1, k_2 \geq 0$.

3.) $L_1 := \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ works;

$L_2 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ does not work.