# Series 9:

## Task 1:

To show that $R^\sim$ is indeed a rational matrix write

$$R(\lambda) = \begin{bmatrix} r_{1,1}(\lambda) & \cdots & r_{1,q}(\lambda) \\ \vdots & & \\ r_{p,1}(\lambda) & \cdots & r_{p,q}(\lambda) \end{bmatrix}$$

with $r_{i,j}(\lambda) \in \mathbb{C}(\lambda)$. Then

$$R^\sim(\lambda) = R^*(-\bar{\lambda}) = \begin{bmatrix} \overline{r_{1,1}(-\bar{\lambda})} & \cdots & \overline{r_{p,1}(-\bar{\lambda})} \\ \vdots & & \\ \overline{r_{1,q}(-\bar{\lambda})} & \cdots & \overline{r_{p,q}(-\bar{\lambda})} \end{bmatrix}$$

and it is sufficient to show that with $r \in \mathbb{C}(\lambda)$ also $r^\sim \in \mathbb{C}(\lambda)$. Therefore write

$$r(\lambda) = \frac{a_k \lambda^k + \ldots + a_1 \lambda + a_0}{b_\ell \lambda^\ell + \ldots + b_1 \lambda + b_0}$$

and observe that

$$r^\sim(\lambda) = \overline{r(-\bar{\lambda})} = \overline{\left( \frac{a_k (-\bar{\lambda})^k + \ldots + a_1(-\bar{\lambda}) + a_0}{b_\ell (-\bar{\lambda})^\ell + \ldots + b_1(-\bar{\lambda}) + b_0} \right)}$$

$$= \frac{(-1)^k \overline{a_k} \lambda^k + \ldots + (-1)^1 \overline{a_1} \lambda + \overline{a_0}}{(-1)^\ell \overline{b_\ell} \lambda^\ell + \ldots + (-1)^1 \overline{b_1} \lambda + \overline{b_0}}$$

is the ratio of two polynomials in $\lambda$.

1a) $$\left(B(\lambda)C(\lambda)\right)^\sim = \left(B(-\bar{\lambda})\, C(-\bar{\lambda})\right)^*$$
$$= C^*(-\bar{\lambda})\, B^*(-\bar{\lambda}) = C^\sim(\lambda)\, B^\sim(\lambda)$$

2.) If $AA^{-1} = I$ then by 1.) we have

$$I = I^\sim = (AA^{-1})^\sim = (A^{-1})^\sim A^\sim$$

$\Rightarrow (A^{-1})^\sim$ is inverse of $A^\sim$

$$\Rightarrow (A^\sim)^{-1} = (A^{-1})^\sim$$

3.) $$\left(B(\lambda)^\sim\right)^\sim = \left(B(-\bar{\lambda})^*\right)^\sim = \left(B\left(-\overline{(-\bar{\lambda})}\right)^*\right)^*$$

$$= B(\lambda)$$

4.) $$\left(B^\sim A B\right)^\sim \overset{1.)}{=} B^\sim A^\sim (B^\sim)^\sim \overset{3.)}{=} B^\sim A B$$

5.) If $\tilde{U} \in \mathbb{C}[\lambda]^{p,p}$ satisfies $I = U\tilde{U} = U\tilde{U}$ then by 1.) also

$$I = I^\sim = (U\tilde{U})^\sim = (\tilde{U}U)^\sim$$

$$= (\tilde{U}^\sim)(U^\sim) = (U^\sim)(\tilde{U}^\sim)$$

which means that $\tilde{U}^\sim$ is a polynomial inverse of $U^\sim$.

6.) If $B = S\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$, $D \in \mathbb{C}(\lambda)^{r,r}$ is the MacMillan form of $B$ then

$$B^\sim \overset{1.)}{=} T^\sim \begin{bmatrix} D^\sim & 0 \\ 0 & 0 \end{bmatrix} S^\sim, \quad D^\sim \in \mathbb{C}(\lambda)^{r,r} \text{ is}$$

the MacMillan form of $B^\sim$.

$D$ and $D^{\sim}$ have the same size.

7.) Follows from 6.)

## Task 2:

$$P^{\sim}(\lambda) = P(-\bar{\lambda})^* = \left( \sum_{i=0}^{k} (-\bar{\lambda})^i P_i \right)^*$$

$$= \sum_{i=0}^{k} \left( (-\bar{\lambda})^i P_i \right)^* = \sum_{i=0}^{k} \overline{(-\bar{\lambda})^i}\, P_i^*$$

$$= \sum_{i=0}^{k} \lambda^i \left[ (-1)^i P_i^* \right] .$$

## Task 3:

We are looking for an $H = H^*$ such that
$\hat{z}^* H \hat{z} = \hat{z}^* \mathcal{M} \hat{z}$ for all $\hat{z} \in \mathbb{R}^{kq}$.

We have

$$\hat{z}^* \mathcal{M} \hat{z} \overset{\hat{z} \in \mathbb{R}^{kq}}{\underset{\downarrow}{=}} \hat{z}^T \mathcal{M} \hat{z} = \frac{1}{2} \left( \hat{z}^T \mathcal{M} \hat{z} + \underbrace{\left( \hat{z}^T \mathcal{M} \hat{z} \right)^T}_{\in \mathbb{R}} \right)$$

$$= \frac{1}{2} \left( \hat{z}^T \mathcal{M} \hat{z} + \hat{z}^T \mathcal{M}^T \hat{z} \right) = \hat{z}^T \underbrace{\left( \frac{1}{2} (\mathcal{M} + \mathcal{M}^T) \right)}_{=: H} \hat{z}$$

where $H^* \overset{H \in \mathbb{R}^{qk, qk}}{\underset{\downarrow}{=}} H^T = \frac{1}{2} (\mathcal{M} + \mathcal{M}^T)^T = \dots = H$

~~Task 4:~~

# Task 4:

1.) By assumption there exist constants $\alpha_i, \beta_i > 0$ such that

$$\left| f^{(i)}(t) \right| \leq \alpha_i e^{-\beta_i t},$$

for all $t \geq 0$. We distinguish two cases. First assume that $\operatorname{Re}(a) \geq 0$. Define

$$x_0 := -\int_0^\infty e^{-as} f(s) ds.$$

Note that $x_0$ is well defined, since in this case $e^{-as}$ is bounded for all $s \geq 0$ by 1 and $f$ is exponentially decaying and infinitely often differentiable. With the variation-of-constants formula and $x_0$ as an initial condition we obtain that

$$\begin{aligned}
x(t) &= e^{at} x_0 + e^{at} \int_0^t e^{-as} f(s) ds \\
&= e^{at} \left( -\int_0^\infty e^{-as} f(s) ds + \int_0^t e^{-as} f(s) ds \right) \\
&= -e^{at} \int_t^\infty e^{-as} f(s) ds
\end{aligned}$$

is a solution of $\dot{x} = ax + f$. We have

$$\begin{aligned}
|x(t)| &= \left| e^{at} \right| \left| \int_t^\infty e^{-as} f(s) ds \right| \\
&\leq e^{\operatorname{Re}(a)t} \int_t^\infty \left| e^{-as} f(s) \right| ds \\
&\leq e^{\operatorname{Re}(a)t} \int_t^\infty e^{-\operatorname{Re}(a)s} \alpha_0 e^{-\beta_0 s} ds \\
&= \alpha_0 e^{\operatorname{Re}(a)t} \int_t^\infty e^{-(\operatorname{Re}(a)+\beta_0)s} ds \\
&= \alpha_0 e^{\operatorname{Re}(a)t} \left( -\frac{1}{\operatorname{Re}(a) + \beta_0} e^{-(\operatorname{Re}(a)+\beta_0)s} \Big|_t^\infty \right) \\
&= \frac{\alpha_0}{\operatorname{Re}(a) + \beta_0} e^{\operatorname{Re}(a)t} e^{-(\operatorname{Re}(a)+\beta_0)t} \\
&= \frac{\alpha_0}{\operatorname{Re}(a) + \beta_0} e^{-\beta_0 t} =: \gamma_0 e^{-\delta_0 t},
\end{aligned}$$

by setting $\gamma_0 := \frac{\alpha_0}{\operatorname{Re}(a)+\beta_0}$ and $\delta_0 := \beta_0$. Using $\dot{x} = ax + f$ repeatedly, we see that

$$x^{(i)}(t) = a^i x(t) + \sum_{j=0}^{i-1} a^j f^{(i-1-j)}(t).$$

Using the vector space structure of $\mathcal{C}_+^1$, this shows that all derivatives of $x$ are also bounded by an exponential function, which implies $x \in \mathcal{C}_+^1$.

If $\operatorname{Re}(a) < 0$ then multiple solutions exist. We choose $x_0 := 0$ and observe that in this case

$$x(t) = \int_0^t e^{a(t-s)} f(s) ds,$$

is a solution of $\dot{x} = ax + f$. W.l.o.g. we assume that $\beta_0 < -\operatorname{Re}(a)$ (otherwise chose $\beta_0$ smaller, which is still appropriate). Then $\beta_0 + \mathrm{Re}\,(a) < 0$ and we have

$$
\begin{aligned}
|x(t)| &\leq \int_0^t \left| e^{a(t-s)} f(s) \right| ds \\
&\leq \int_0^t e^{\mathrm{Re}(a)(t-s)} \alpha_0 e^{-\beta_0 s} ds \\
&= \alpha_0 e^{\mathrm{Re}(a)t} \int_0^t e^{-s(\mathrm{Re}(a)+\beta_0)} ds \\
&= -\frac{\alpha_0}{\mathrm{Re}\,(a) + \beta_0} e^{\mathrm{Re}(a)t} \left( e^{-s(\mathrm{Re}(a)+\beta_0)} \Big|_0^t \right) \\
&= -\frac{\alpha_0}{\mathrm{Re}\,(a) + \beta_0} e^{\mathrm{Re}(a)t} \left( e^{-t(\mathrm{Re}(a)+\beta_0)} - 1 \right) \\
&= -\frac{\alpha_0}{\mathrm{Re}\,(a) + \beta_0} \left( e^{-\beta_0 t} - e^{\mathrm{Re}(a)t} \right) =: \gamma_0 \left( e^{-\delta_0 t} - e^{\mathrm{Re}(a)t} \right) \leq \gamma_0 e^{-\delta_0 t},
\end{aligned}
$$

for all $t \geq 0$. As before we deduce that then all derivatives of $y$ are also bounded by an exponential function, since $y$ solves $\dot{x} = ax + f$, and thus $y \in \mathcal{C}_+^1$.

2.) Using the Jordan canonical form of A the problem decomposes into a finite number of subproblems of which each has the form

$$
\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{n_i} \end{bmatrix} = \begin{bmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_i} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n_i} \end{bmatrix},
$$

with $n_i \in \mathbb{N}$. Starting from the last variable and last equation one can use 1.) to show that there exists a $y_{n_i} \in \mathcal{C}_+^1$ such that the last equation is fulfilled. Using the fact that $\mathcal{C}_+^1$ is a vector space we see that $y_{n_i} + f_{n_i-1} \in \mathcal{C}_+^1$. Thus, using 1.) again, we find that there exists a $y_{n_i-1} \in \mathcal{C}_+^1$ such that also the second last equation is fulfilled. Proceeding this way we obtain the claim for the subproblem and thus also for an arbitrary matrix $A$.

3.) We distinguish two cases. For the first case assume that $d \equiv c \in \mathbb{C} \setminus \{0\}$ is a non-zero constant. In this case we set $x := \frac{1}{c} b$ and obtain the assertion immediately. For the second case let $d(\lambda) = \sum_{i=0}^K \alpha_i \lambda^i$ with $K \in \mathbb{N}$ and $d_K \neq 0$. This means that we are looking for a solution $x \in \mathcal{C}_+^1$ of the differential equation $\sum_{i=0}^K \alpha_i x^{(i)}(t) = b(t)$. Reducing this higher order, scalar differential equation to a first order, matrix differential equation (as done in basic courses about ODEs), and applying 2.) we immediately obtain the result.

4.) Let a Smith-form of $P$ be given by $P = SDT$ where $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ are unimodular and $D \in \mathbb{C}[\lambda]^{p,q}$ is diagonal of the form

$$
D = \begin{bmatrix} d_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & d_p & 0 & \cdots & 0 \end{bmatrix},
$$

with $d_1, \ldots, d_p \in \mathbb{C}[\lambda] \setminus \{0\}$. Set $\tilde{b} := S^{-1}\left(\frac{d}{dt}\right) b$. Then we see that $\tilde{b} \in \mathcal{C}_+^p$ is itself exponentially decaying. For $i = 1, \ldots, p$ denote the elements of $\tilde{b}$ by $\tilde{b}_i$ and construct $x_i \in \mathcal{C}_+^1$ with 3.) as exponentially decaying solutions of the scalar equations $d_i\left(\frac{d}{dt}\right) x_i = \tilde{b}_i$. Define $\tilde{x} \in \mathcal{C}_+^q$ through

$$
\tilde{x} := \begin{bmatrix} x_1 & \cdots & x_p & 0 \cdots & 0 \end{bmatrix}^T
$$

and notice that this implies $D\left(\frac{d}{dt}\right) \tilde{x} = \tilde{b}$. Thus, setting $x := T^{-1}\left(\frac{d}{dt}\right) \tilde{x} \in \mathcal{C}_+^q$ proves the claim.

## Task 5:

In this case $\mathfrak{B}_+(P)$ is autonomous and thus for $z_0 \in \mathfrak{B}_+(P)$ we have

$$\inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t) = z_0(t),\, t \leq 0}} \int_0^\infty \dots \, dt = \int_0^\infty (\Delta z_0(t))^* H (\Delta z_0(t)) \, dt .$$

The set over which the infimum is taken only contains one element: $z_0$

To show non-negativity let $z \in \mathfrak{B}_+(P)$ with $z(t) = 0$, $t \leq 0$. Since $\mathfrak{B}_+$ is autonomous this implies that $z = 0$ and thus

$$\int_0^\infty (\Delta z)^* H (\Delta z) \, dt = \int_0^\infty 0 \, dt = 0 \geq 0 .$$

## Task 6:

If $H$ is not non-negativ w.r.t. $P$ then there exists a $v \in \mathfrak{B}_+(P)$ with $v(t) = 0$, $t \leq 0$ and

$$0 > \int_0^\infty (\Delta v)^* H (\Delta v) \, dt =: c$$

Setting $\hat{z} := z_0 + \alpha v \in \mathfrak{B}_+(P)$ then shows that for $t \leq 0$: $\hat{z}(t) = z_0(t) + \alpha \underbrace{v(t)}_{=0} = z_0(t)$

while

$$\int_0^\infty (\Delta \hat{z})^* H (\Delta \hat{z}) \, dt = \underbrace{\int_0^\infty (\Delta z_0)^* H (\Delta z_0) \, dt}_{=: a}$$

$$+\alpha\underbrace{\left(2\operatorname{Re}\left\{\int_0^\infty (\Delta z_0)^* H(\Delta v)\,dt\right\}\right)}_{=:b} + \alpha^2 \underbrace{\int_0^\infty (\Delta v)^* H(\Delta v)\,dt}_{=c}$$

$$= a + \alpha b + \alpha^2 c.$$

Since $c<0$ one can choose $\alpha$ such that $a+\alpha b+\alpha^2 c$ becomes arbitrarily small.

## Task 7:

Choose $P(\lambda) = -\lambda \in \mathbb{C}[\lambda]^{1,1}$. Then

$P^\sim(\lambda) = -(-\bar{\lambda})^* = \lambda$ and

$P^{\langle 1\rangle}(\lambda) = -1$, $P^{\langle \ell\rangle}(\lambda) = 0$, $\ell \geq 2$.

Thus Lemma 4.2 reads

$$\int_{t_0}^{t_1} \overline{z(t)}\,\dot{y}(t)\,dt = \int_{t_0}^{t_1} z^*(t)\left(P^\sim(\tfrac{d}{dt})y(t)\right)dt$$

$$= \int_{t_0}^{t_1}\left(P(\tfrac{d}{dt})z(t)\right)^* y(t)\,dt + \sum_{\ell=1}^{1}(-1)^\ell \left(P^{\langle \ell\rangle}(\tfrac{d}{dt})z(t)\right)^* y^{(\ell-1)}(t)\Big|_{t_0}^{t_1}$$

$$= -\int_{t_0}^{t_1} \dot{z}^*(t)\,y(t)\,dt + (-1)^1\left(-z(t)\right)^* y(t)\Big|_{t_0}^{t_1}$$

$$= \overline{z(t)}\,y(t)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{\overline{z(t)}}\,y(t)\,dt.$$

Replacing $z$ by $\bar{z}$ yields the result.

Task 8: Let $\|z^{(i)}(t)\| \le \alpha_i e^{-\beta_i t}$, $P(\lambda) = \sum_{i=0}^{k} \lambda^i P_i$

(a) For $j \in \mathbb{N}_0$ we have for $t \ge 0$

$$\left\|\left(\tfrac{d}{dt}\right)^j P\left(\tfrac{d}{dt}\right) z\right\| = \left\| \sum_{i=0}^{k} P_i \, z^{(i+j)}(t) \right\|$$

$$\le \sum_{i=0}^{k} \|P_i\| \cdot \|z^{(i+j)}(t)\|$$

$$\le \sum_{i=0}^{k} \|P_i\| \cdot \alpha_{i+j} \, e^{-\beta_{i+j} t} .$$

Setting $\tilde{\beta}_j := \min\{\beta_0, \beta_1, \dots, \beta_{j+k}\}$ we have

$$e^{-\beta_\ell t} \le e^{-\tilde{\beta}_j t} \quad \text{for } t \ge 0$$

and $\ell \le j+k$.

$$\Rightarrow \left\|\left(\tfrac{d}{dt}\right)^j P\left(\tfrac{d}{dt}\right) z(t)\right\| \le \underbrace{\left( \sum_{i=0}^{k} \|P_i\| \cdot \alpha_{i+j} \right)}_{=:\, \tilde{\alpha}_j} \cdot e^{-\tilde{\beta}_j t}$$

$$\Rightarrow P\left(\tfrac{d}{dt}\right) z \in \mathcal{C}^{p}_{\infty} .$$

(b) If $z(t) = 0$ for $t \le 0$ then also $z^{(j)}(t) = 0$ for $t \le 0$ and $j \in \mathbb{N}_0$.

Thus, for $t \le 0$

$$P\left(\tfrac{d}{dt}\right) z(t) = \sum_{i=0}^{k} P_i \underbrace{z^{(i)}(t)}_{=0} = 0 .$$

# Task 9:

1.) If $q, f$ are real we have

$$-\int_0^\infty \mathrm{Re}(\dot{q}(t) f(t))\,dt = \int_0^\infty \dot{q}(t) f(t)\,dt$$

$$= \int_0^\infty \begin{bmatrix} q \\ f \\ \dot{q} \\ \dot{f} \end{bmatrix}^T \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & & \end{array}\right] \begin{bmatrix} q \\ f \\ \dot{q} \\ \dot{f} \end{bmatrix} dt = \int_0^\infty \begin{bmatrix} q \\ f \\ \vdots \end{bmatrix}^* \underbrace{\left[\begin{array}{cc|cc} & & 0 & 0 \\ & & \frac{1}{2} & 0 \\ \hline 0 & \frac{1}{2} & & \\ 0 & 0 & & \end{array}\right]}_{=:H} \begin{bmatrix} q \\ f \\ \vdots \end{bmatrix} dt.$$

In general, if $q, f$ are complex we still have

$$\sup_{\dots} -\int_0^\infty \mathrm{Re}(\dot{q}(t) \cdot f(t))\,dt$$

$$= -\inf_{\dots} \int_0^\infty \mathrm{Re}(\dot{q}(t) \cdot f(t))\,dt = -\inf_{\dots} \int_0^\infty \left(\Delta_2 \begin{bmatrix} q \\ f \end{bmatrix}\right)^* H \left(\Delta_2 \begin{bmatrix} q \\ f \end{bmatrix}\right) dt.$$

Since we have

$$(\Delta_2^2(\lambda))^\sim H (\Delta_2^2(\lambda)) = (\Delta_2^2(\lambda))^\sim \left[\begin{array}{cc|cc} & & & 0 \\ & & \frac{1}{2} & \\ \hline & \frac{1}{2} & & \\ 0 & & & \end{array}\right] \begin{bmatrix} 1 & \\ & 1 \\ \lambda & \\ & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & -\lambda & \\ & 1 & & -\lambda \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\lambda}{2} \\ \frac{\lambda}{2} & 0 \end{bmatrix}$$

the optimality system is

$$\mathscr{L}_+\left(\begin{bmatrix} & P \\ P^\sim & (\Delta_2^2)^\sim H (\Delta_2^2) \end{bmatrix}\right) = \mathscr{L}_+\left(\left[\begin{array}{c|c|c} 0 & m\lambda^2 + d\lambda + k & -1 \\ \hline \overline{m}\lambda^2 - \overline{d}\lambda + k & 0 & -\frac{\lambda}{2} \\ -1 & \frac{\lambda}{2} & 0 \end{array}\right]\right).$$

2.) We have

$$\inf_{\cdots} \int_0^\infty \left(|\dot{q}(t)|^2 + |q(t)|^2\right) dt$$

$$= \inf_{\cdots} \int_0^\infty \begin{bmatrix} q \\ f \\ \dot{q} \\ \dot{f} \end{bmatrix}^* \underbrace{\left[\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & 1 & \\ & & & 0 \end{array}\right]}_{=:H} \begin{bmatrix} q \\ f \\ \dot{q} \\ \dot{f} \end{bmatrix} dt = \inf_{\cdots} \int_0^\infty \left(\Delta_2 \begin{bmatrix} q \\ f \end{bmatrix}\right)^* H \left(\Delta_2 \begin{bmatrix} q \\ f \end{bmatrix}\right) dt$$

and since

$$(\Delta_2^2(\lambda))^\sim H \Delta_2^2(\lambda) = (\Delta_2^2(\lambda))^\sim \left[\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & 1 & \\ & & & 0 \end{array}\right] \begin{bmatrix} 1 & \\ & 1 \\ \lambda & \\ & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & -\lambda & \\ & 1 & & -\lambda \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \lambda & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1-\lambda^2 & 0 \\ 0 & 0 \end{bmatrix}$$

the optimality system is

$$\mathcal{L}_+ \left( \left[\begin{array}{c|c|c} 0 & m\lambda^2 + d\lambda + k & -1 \\ \hline \bar{m}\lambda^2 - \bar{d}\lambda + \bar{k} & 1-\lambda^2 & 0 \\ -1 & 0 & 0 \end{array}\right] \right)$$

3.) As in 2.)

$$H := \begin{bmatrix} 1/2 & & & \\ & \varepsilon/2 & & \\ & & 1/2 & \\ & & & 0 \end{bmatrix}$$ and

the optimality system is

$$\mathcal{L}_+ \left( \left[\begin{array}{c|c|c} 0 & m\lambda^2 + d\lambda + k & 1 \\ \hline \bar{m}\lambda^2 - \bar{d}\lambda + \bar{k} & 1-\lambda^2 & 0 \\ -1 & 0 & \varepsilon/2 \end{array}\right] \right)$$

# Task 10:

As in Task 9 we have

$$\sup_{\dots} -\int_0^\infty \mathrm{Re}\,(V(f)\,\bar{I}(f))\,df = -\inf_{\dots}\int_0^\infty \mathrm{Re}\,(V(f)\,\bar{I}(f))\,df$$

$$= -\inf_{\dots}\int_0^\infty z^*(f)\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}}_{=:H} z(f)\,df$$

and since

$$\underbrace{\Delta_1^4(\lambda)^\sim}_{=I} H \underbrace{\Delta_1^4(\lambda)}_{=I} = H$$

the optimality system is

$$\mathcal{L}_+\left(\left[\begin{array}{ccc|cccc} & & & 1 & 1 & 1 & 0 \\ & & & -R & 0 & 0 & 1 \\ & & & 0 & \lambda L & 0 & -1 \\ \hline 1 & -R & 0 & 0 & & & \\ 1 & 0 & -\lambda L & & 0 & & \\ 1 & 0 & 0 & & & 0 & \frac{1}{2} \\ 0 & 1 & -1 & & & \frac{1}{2} & 0 \end{array}\right]\right).$$

$$(R, L \in \mathbb{R} \Rightarrow \bar{R} = R, \bar{L} = L)$$