Series 12:

Task 1:

(a)
$$0 = Q - A^*X - X^*A - (S - X^*B)R^{-1}(S^* - B^*X)$$
$$= Q - A^*X - X^*A - \left[SR^{-1}S^* - SR^{-1}B^*X - X^*BR^{-1}S^* + X^*BR^{-1}B^*X \right]$$
$$= \underbrace{Q - SR^{-1}S}_{=:-H} + X^*\underbrace{(BR^{-1}S^* - A)}_{=:F} + \underbrace{(SR^{-1}B^* - A^*)}_{=F^*}X - X^*\underbrace{BR^{-1}B^*}_{=:-G}X .$$

(b)
$$\left[\begin{bmatrix} F & G \\ H & -F^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X^* \end{bmatrix} (F + GX) \right] \Leftrightarrow$$
$$\left[\begin{matrix} F + GX = F + GX \\ H - F^*X = X^*F + X^*GX \end{matrix} \right] \Leftrightarrow$$
$$\left[H - F^*X = X^*F + X^*GX \right] \Leftrightarrow$$
$$\left[0 = X^*GX + X^*F - F^*X - H \right]$$

(c) Let $\lambda_0 \in \sigma(F+GX)$ and let $v_0 \neq 0$ be an associated eigenvector

$$(F+GX)v_0 = \lambda_0 v_0$$

Then

$$\mathcal{H} \underbrace{\begin{bmatrix} I \\ X \end{bmatrix} v_0}_{=: w_0 \neq 0} = \underbrace{\begin{bmatrix} I \\ X \end{bmatrix}}_{= \begin{bmatrix} I \\ X \end{bmatrix}} \underbrace{(F+GX)v_0}_{= \lambda_0 v_0} = \lambda_0 \begin{bmatrix} I \\ X \end{bmatrix} v_0 = \lambda_0 w_0$$

$$\Rightarrow \quad \lambda_0 \in \sigma(\mathcal{H}).$$

Task 2:

(a) If $n=1$ and $A =: a$, $B =: b$, ... (1)

~~test~~ reads

$$0 = q - \bar{a}x - \bar{x}a - (s - \bar{x}b)r(\bar{s} - \bar{b}x)$$

$$= q - sr\bar{s} + \bar{x}(br\bar{s} - \bar{a}) + (sr\bar{b} - \bar{a})x$$

$$- \bar{x}br\bar{b}x.$$

Since $q = Q = Q^* = \bar{q}$ is real (and similar also $x, r \in \mathbb{R}$ are real) this is equivalent to

$$0 = \underbrace{(q - |s|^2 r)}_{=: +h \in \mathbb{R}} + x \cdot \underbrace{2\,\mathrm{Re}(sr\bar{b} - \bar{a})}_{=: f \in \mathbb{R}} + x^2 \cdot \underbrace{|b|^2 r}_{=: g \in \mathbb{R}}$$

$$= h + fx + gx^2$$

$\Rightarrow$ The solutions are given by

$$x_{1,2} = \frac{1}{2g}\left(-f \pm \sqrt{f^2 - 4gh}\right)$$

(b)

1.) For $\Delta \in H^n$ we have

$$\mathcal{R}(X+\Delta) - \mathcal{R}(X)$$

$$= (X+\Delta)^* G (X+\Delta) + F^*(X+\Delta) + (X+\Delta)^* F - H$$

$$- \mathcal{R}(X)$$

$$= \cancel{X^\circ G X} + \Delta^\circ G X + X^\circ G \Delta + \Delta^\circ G \Delta$$

$$+ \cancel{F^\circ X} + F^\circ \Delta + \cancel{X^\circ F} + \Delta^\circ F \cancel{- H}$$

$$\cancel{- X^\circ G X} \cancel{- F^\circ X} \cancel{- X^\circ F} \cancel{+ H}$$

$$= \Delta^\circ (F + GX) + (F^\circ + X^\circ G^\circ) \Delta + \Delta^\circ G \Delta$$

and thus we choose the linear term

$$(*) \quad D\mathcal{R}(X)[\Delta] := \Delta^\circ (F + GX) + (F + GX)^\circ \Delta$$

to see that

$$\lim_{\Delta \to 0} \frac{1}{\|\Delta\|} \| \mathcal{R}(X + \Delta) - \mathcal{R}(X) - D\mathcal{R}(X)[\Delta] \|$$

$$= \lim_{\Delta \to 0} \frac{1}{\|\Delta\|} \| \Delta^\circ G \Delta \| \leq \lim_{\Delta \to 0} \frac{1}{\|\Delta\|} \cdot \|\Delta^\circ\| \cdot \|G\| \cdot \|\Delta\|$$

$$= \lim_{\Delta \to 0} \|G\| \cdot \|\Delta\| = 0 ,$$

which means that $(*)$ is indeed the derivative.

2.) By definition $(D\mathcal{R}(X))^{-1}[\Delta]$ is the matrix $\underline{\Delta} \in H^n$ which fulfills

$$D\mathcal{R}(X)[\underline{\Delta}] = \Delta$$

$$\Leftrightarrow \underline{\Delta}^* \underbrace{(F+GX)}_{=:A} + (F+GX)^* \underline{\Delta} = \Delta$$

$\Rightarrow$ Solve the Lyapunov equation

$$\underline{\Delta}^* A + A^* \underline{\Delta} = \Delta$$

for $\underline{\Delta}$.

3.) The Newton step is

$$X_{k+1} = X_k - (D\mathcal{R}(X_k))^{-1}[\mathcal{R}(X_k)].$$

Thus, for one iteration we have to

i) compute $\mathcal{R}(X_k) = X_k^* G X_k + \dots$

ii) solve the Lyapunov equation

$$\underline{\Delta}^* (F+GX_k) + (F+GX_k)^* \underline{\Delta} = \mathcal{R}(X_k)$$

iii) update $X_{k+1} = X_k - \underline{\Delta}$.

Task 3:

(a) For $\eta := (x, \mathring{x}, u) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^m$

Taylor expansion yields

$$F(x,\mathring{x},u) = F(\eta) = \underbrace{F(\eta_0)}_{=0} + DF(\eta_0)(\eta-\eta_0) + O(\|\eta-\eta_0\|^2)$$

$$\approx DF(\eta_0)(\eta-\eta_0)$$

$$= \left[\underbrace{\tfrac{\partial}{\partial x}F(\eta_0)}_{=:A}, \underbrace{\tfrac{\partial}{\partial \mathring{x}}F(\eta_0)}_{=:-E}, \tfrac{\partial}{\partial u}F(\eta_0)\right]\begin{bmatrix} x-x_0 \\ \mathring{x} \\ u-u_0 \end{bmatrix}$$

which gives for $x \in \mathcal{C}_\infty^n$, $u \in \mathcal{C}_\infty^m$ that

$$0 \approx Ax(t) - Ax_0 - E\mathring{x}(t) + Bu(t) - Bu_0$$

$$\Leftrightarrow \quad E\mathring{x}(t) \approx Ax(t) + Bu(t) + [-Ax_0 - Bu_0].$$

Defining $\tilde{x}(t) := x(t) - x_0$, $\tilde{u}(t) := u(t) - u_0$ as the deviation from the steady state this corresponds to the linear system

$$E\mathring{\tilde{x}}(t) \approx A\tilde{x}(t) + B\tilde{u}(t)$$

with behavior

$$\mathfrak{L}([\lambda E-A, -B]).$$

For $(z, \dot{z}) \in \mathbb{C}^q \times \mathbb{C}^q$ Taylor gives

$$F(z, \dot{z}) = \underbrace{F(z_0, 0)}_{=0} + DF(z, 0) \begin{bmatrix} z - z_0 \\ \dot{z} \end{bmatrix} + \mathcal{O}\left(\left\| \begin{bmatrix} z - z_0 \\ \dot{z} \end{bmatrix} \right\|^2\right)$$

$$\approx \Big[\underbrace{\tfrac{\partial}{\partial z} F(z, 0)}_{=: G}, \underbrace{\tfrac{\partial}{\partial \dot{z}} F(z, 0)}_{=: F} \Big] \begin{bmatrix} z - z_0 \\ \dot{z} \end{bmatrix}$$

which for $z \in \mathcal{C}^q_\infty$ implies

$$0 \approx F \dot{z}(t) + G(z(t) - z_0)$$

and with the deviation $\tilde{z}(t) := z(t) - z_0$

$$F \dot{\tilde{z}}(t) + G \tilde{z}(t) = 0$$

(c) With $z := (x, u)$ we have

$$0 = F(x(t), \dot{x}(t), u(t)) = \tilde{F}(z(t), \dot{z}(t))$$

by setting $\tilde{F} : \mathbb{C}^{n+m} \times \mathbb{C}^{n+m} \to \mathbb{C}^p$ through

$$\tilde{F}(z, \dot{z}) = \tilde{F}(x, u, \dot{x}, \dot{u}) := F(x, \dot{x}, u).$$

(d) For $\underbrace{(z, \dot{z}, \ldots, z^{(k)})}_{=: \eta} \in \mathbb{C}^{(k+1)q}$ Taylor gives

$$F(\eta) = \underbrace{F(\eta_0)}_{=0} + DF(\eta_0)\,(\eta - \eta_0) + \mathcal{O}(\|\eta - \eta_0\|^2)$$

$$\approx \left[\underbrace{\frac{\partial}{\partial z} F(\eta_0)}_{=: P_0}, \underbrace{\frac{\partial}{\partial \dot{z}} F(\eta_0)}_{=: P_1}, \ldots, \underbrace{\frac{\partial}{\partial z^{(k)}} F(\eta_0)}_{=: P_k} \right] \begin{bmatrix} z - z_0 \\ \dot{z} \\ \vdots \\ z^{(k)} \end{bmatrix}$$

which for $z \in \mathcal{C}_\infty^q$ implies

$$0 \approx P_0 (z(t) - z_0) + P_1 \dot{z}(t) + \ldots + P_k z^{(k)}(t)$$

and with the deviation $\tilde{z}(t) := z(t) - z_0$

$$0 = P_0 \tilde{z}(t) + \ldots + P_k \tilde{z}^{(k)}(t) = \sum_{i=0}^{k} P_i \tilde{z}^{(i)}(t)$$

which corresponds to the behavior

$$\mathfrak{B}\left(\sum_{i=0}^{k} P_i \lambda^i \right).$$