Constant controllers for first order systems

To store $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ in a computer one needs (at most) $2 \cdot p \cdot q$ doubles of memory. If one computes a regular, stabilizing controller $C \in \mathbb{C}[\lambda]_d^{c,q}$ of polynomial degree $d \in \mathbb{N}_0$ (as it was constructed in the previous section "Stabilization") one would in general need

$$((d+1) \cdot c \cdot q) \text{ doubles},$$

where we know nothing about d. This could take a lot of memory if d is large.

Here we will show that for controllable first order systems, i.e., systems of the form $\mathfrak{B}(\lambda F + G)$ with $\mathfrak{Z}(\lambda F + G) = \emptyset$, one can construct regular, stabilizing controllers which are constant matrices, i.e., $C \in \mathbb{C}^{c,q} = \mathbb{C}[\lambda]_0^{c,q}$. The construction is based on the Kronecker canonical form. Therefore, we first construct constant, regular, stabilizing controllers for each of the blocks in the Kronecker canonical form separately (if possible).

First consider the blocks of type $\mathcal{J}, \mathcal{N}, \mathcal{M}$. Since all blocks of this type have full column rank the systems $\mathfrak{B}(\mathcal{J}), \mathfrak{B}(\mathcal{N}), \mathfrak{B}(\mathcal{M})$ are already autonomous (Theorem 1.19) and one can show (Homework, Series 7, Task 8) that for these systems there exist no regular controllers.

Regular controllers can only be constructed for the blocks of type $\mathcal{L}$. Thus consider a block of this type of size $\epsilon_j \in \mathbb{N}_0$, i.e.,

$$\mathcal{L}_{\epsilon_j}(\lambda) := \lambda L_{1,j} - L_{0,j} := \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{C}[\lambda]_1^{\epsilon_j, \epsilon_j+1}. \tag{1}$$

Let $p(\lambda) = \lambda^{\epsilon_j} c_{\epsilon_j} + \ldots + \lambda c_1 + c_0 \in \mathbb{C}[\lambda]_{\epsilon_j}$ be a polynomial of degree $\leq \epsilon_j$ and define the matrix $C := \begin{bmatrix} c_0 & \cdots & c_{\epsilon_j} \end{bmatrix} \in \mathbb{C}^{1,\epsilon_j+1}$. Then one can show that

$$\begin{aligned} \mathfrak{Z}\left(\begin{bmatrix} \mathcal{L}_{\epsilon_j}(\lambda) \\ C \end{bmatrix}\right) &= \mathfrak{Z}\left(\begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \\ c_0 & \cdots & c_{\epsilon_j-1} & c_{\epsilon_j} \end{bmatrix}\right) = \mathfrak{Z}\left(p(\lambda)\right) \qquad \text{and} \\ \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \mathcal{L}_{\epsilon_j} \\ C \end{bmatrix}\right) &= \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\mathcal{L}_{\epsilon_j}\right) + \operatorname{rank}_{\mathbb{C}(\lambda)}(C) = \epsilon_j + \operatorname{rank}(C) \end{aligned}$$

(Homework, Series 8, Task 1).

This implies that C is a regular controller for $\mathfrak{B}(\mathcal{L}_{\epsilon_j})$ and (by Theorem 1.19) that $\mathfrak{B}\left(\begin{bmatrix} \mathcal{L}_{\epsilon_j}(\lambda) \\ C \end{bmatrix}\right)$ is autonomous if and only if $C \neq 0$ which is the case if and only if $p \neq 0$.

Thus, to construct a regular, stabilizing controller for $\mathfrak{B}(\mathcal{L}_{\epsilon_j})$ one can simply pick a stable $p \neq 0$ (which means $\mathfrak{Z}(p) \subset \mathbb{C}_-$) of degree $\leq \epsilon_j$, for example

$$p(\lambda) = (\lambda + 42)^{\epsilon_j} \quad \text{or} \quad p(\lambda) = (\lambda + 1.2345) \quad \text{or} \quad p(\lambda) = (\lambda+1)(\lambda+2)\cdots(\lambda+\epsilon_j), \tag{2}$$

then determine the coefficients in the canonical basis (i.e., $1, \lambda, \lambda^2, \ldots$), and finally write these coefficients into a matrix C.

As shown by the first (or last; but not the middle) expression in (2) it is no problem to choose p such that the highest coefficient is non-zero: $c_{\epsilon_j} \neq 0$.

Lemma 1. *For a block of the form $\mathcal{L}_{\epsilon_j}$ as in (1) and given poles $\lambda_1^{(j)}, \ldots, \lambda_{\epsilon_j}^{(j)} \in \mathbb{C}_-$ there exists a constant, regular, stabilizing controller $C_j \in \mathbb{C}^{1,\epsilon_j+1}$ such that*

$$\begin{aligned} \mathfrak{Z}\left(\begin{bmatrix} \mathcal{L}_{\epsilon_j}(\lambda) \\ C_j \end{bmatrix}\right) &= \left\{\lambda_1^{(j)}, \ldots, \lambda_{\epsilon_j}^{(j)}\right\} \qquad \text{and} \\ \operatorname{rank}\left(\begin{bmatrix} L_{1,j} \\ C_j \end{bmatrix}\right) &= \operatorname{rank}(L_{1,j}) + \operatorname{rank}(C_j) = \operatorname{rank}(L_{1,j}) + 1 = \epsilon_j + 1. \end{aligned}$$

Proof. Compute the coefficients $c_0, \ldots, c_{\epsilon_j}$ in the canonical basis (i.e., $1, \lambda, \lambda^2, \ldots$) of the polynomial

$$\sum_{k=1}^{\epsilon_j} (\lambda - \lambda_k^{(j)}) =: \sum_{k=0}^{\epsilon_j} \lambda^k c_k =: p_j(\lambda),$$

and go through the construction shown above. Then we also see that $c_{\epsilon_j} = 1 \neq 0$ and thus we have

$$\operatorname{rank}\left(\begin{bmatrix} L_{1,j} \\ C_j \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ c_0 & \cdots & c_{\epsilon_j-1} & 1 \end{bmatrix}\right) = \epsilon_j + 1,$$

which proves the claim. □

By applying Lemma 1 to all blocks in the Kronecker canonical form we obtain the following result.

Theorem 2. *Let $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ and consider the associated Kronecker canonical form. Let $\epsilon \in \mathbb{N}_0$ denote the total number of rows in all blocks of type $\mathcal{L}$ together, let $s \in \mathbb{N}_0$ denote the total number of blocks of type $\mathcal{L}$, and let $\rho \in \mathbb{N}_0$ denote the total number of rows (and columns) in all blocks of type $\mathcal{J}$ together, as defined in the statement of the Kronecker canonical form.*

Assume that $\mathfrak{B}(\lambda F + G)$ is stabilizable and let the poles $\lambda_1, \ldots, \lambda_\epsilon \in \mathbb{C}_-$ be given. Then there exists a constant, regular, stabilizing controller $C \in \mathbb{C}^{s,q}$ such that

$$\begin{aligned} \mathfrak{Z}\left(\begin{bmatrix} \lambda F + G \\ C \end{bmatrix}\right) &= \{\lambda_1, \ldots, \lambda_\epsilon\} \cup \mathfrak{Z}(\lambda F + G) \\ \operatorname{rank}\left(\begin{bmatrix} F \\ C \end{bmatrix}\right) &= \operatorname{rank}(F) + \operatorname{rank}(C) = \operatorname{rank}(F) + s \end{aligned} \tag{3}$$

Proof. In this proof we use most of the notation from the statement of the Kronecker canonical form (cf. the handout "First order systems"). Thus, let S, T be invertible such that

$$\lambda F + G = S \cdot \operatorname{diag}(\mathcal{L}_{\epsilon_1}(\lambda), \ldots, \mathcal{L}_{\epsilon_s}(\lambda), \mathcal{J}(\lambda), \mathcal{N}(\lambda), \mathcal{M}(\lambda)) \cdot T. \tag{4}$$

Since we have $\epsilon = \epsilon_1 + \ldots + \epsilon_s$ we can group the values $\lambda_1, \ldots, \lambda_\epsilon$ into s groups, where the number of elements in each group is given by $\epsilon_1, \ldots, \epsilon_s$. More precisely, introducing the notation $\tilde{\epsilon}_j := \epsilon_1 + \ldots + \epsilon_{j-1}$ for $j \geq 2$ and $\tilde{\epsilon}_1 := 0$ we can define

$$\begin{aligned} \lambda_1^{(1)}, \ldots, \lambda_{\epsilon_1}^{(1)} &:= \lambda_{\tilde{\epsilon}_1+1}, \ldots, \lambda_{\tilde{\epsilon}_1+\epsilon_1}, \\ \lambda_1^{(2)}, \ldots, \lambda_{\epsilon_2}^{(2)} &:= \lambda_{\tilde{\epsilon}_2+1}, \ldots, \lambda_{\tilde{\epsilon}_2+\epsilon_2}, \\ &\vdots \\ \lambda_1^{(s)}, \ldots, \lambda_{\epsilon_s}^{(s)} &:= \lambda_{\tilde{\epsilon}_s+1}, \ldots, \lambda_{\tilde{\epsilon}_s+\epsilon_s}. \end{aligned}$$

Using Lemma 1 one can then construct matrices $C_j \in \mathbb{C}^{1,\epsilon_j+1}$ such that

$$\mathfrak{Z}\left(\begin{bmatrix}\mathcal{L}_{\epsilon_j}(\lambda)\\ C_j\end{bmatrix}\right) = \left\{\lambda_1^{(j)},\ldots,\lambda_{\epsilon_j}^{(j)}\right\} \quad \text{and} \quad \operatorname{rank}\left(\begin{bmatrix}L_{1,j}\\ C_j\end{bmatrix}\right) = \operatorname{rank}(L_{1,j}) + \operatorname{rank}(C_j)\,. \tag{5}$$

for $j = 1,\ldots,s$. Setting

$$C := \left(\begin{bmatrix} C_1 & & & 0 & 0 & 0\\ & \ddots & & \vdots & \vdots & \vdots\\ & & C_s & 0 & 0 & 0\end{bmatrix}\cdot T\right) \in \mathbb{C}^{s,q}$$

with partitioning according to (4) we see that

$$\begin{bmatrix} S^{-1} & \\ & I\end{bmatrix}\left(\begin{bmatrix}\lambda F + G\\ C\end{bmatrix}\right)T^{-1} = \left[\begin{array}{cccccc} \mathcal{L}_{\epsilon_1}(\lambda) & & & & & \\ & \ddots & & & & \\ & & \mathcal{L}_{\epsilon_1}(\lambda) & & & \\ & & & \mathcal{J}(\lambda) & & \\ & & & & \mathcal{N}(\lambda) & \\ & & & & & \mathcal{M}(\lambda) \\ \hline C_1 & & & & & \\ & \ddots & & & & \\ & & C_s & & & \end{array}\right].$$

Permuting the block rows we find that there exists a permutation matrix P such that

$$P\begin{bmatrix} S^{-1} & \\ & I\end{bmatrix}\left(\begin{bmatrix}\lambda F + G\\ C\end{bmatrix}\right)T^{-1} = \begin{bmatrix} \begin{bmatrix}\mathcal{L}_{\epsilon_1}(\lambda)\\ C_1\end{bmatrix} & & & & & \\ & \ddots & & & & \\ & & \begin{bmatrix}\mathcal{L}_{\epsilon_s}(\lambda)\\ C_s\end{bmatrix} & & & \\ & & & \mathcal{J}(\lambda) & & \\ & & & & \mathcal{N}(\lambda) & \\ & & & & & \mathcal{M}(\lambda)\end{bmatrix}. \tag{6}$$

Thus we have

$$\begin{aligned}
&\operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix}\lambda F + G\\ C\end{bmatrix}\right)\\
= &\left(\sum_{k=1}^{s}\operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix}\mathcal{L}_{\epsilon_k}(\lambda)\\ C_k\end{bmatrix}\right)\right) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{J}(\lambda)) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}(\lambda)) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{M}(\lambda))\\
\overset{(5)}{=} &\left(\sum_{k=1}^{s}\operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{L}_{\epsilon_k}(\lambda)) + \operatorname{rank}_{\mathbb{C}(\lambda)}(C_k)\right) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{J}(\lambda)) + \ldots\\
= &\left(\sum_{k=1}^{s}\operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{L}_{\epsilon_k}(\lambda))\right) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{J}(\lambda)) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}(\lambda)) + \operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{M}(\lambda))\\
&+\left(\sum_{k=1}^{s}\operatorname{rank}_{\mathbb{C}(\lambda)}(C_k)\right)\\
= &\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) + \operatorname{rank}_{\mathbb{C}(\lambda)}(C)\,,
\end{aligned}$$

i.e., the controller is regular. By the same type of argument we find that (3) holds.

Finally, since all the blocks on the diagonal of the right matrix in (6) have full column rank, so does $\begin{bmatrix} \lambda F + G \\ C \end{bmatrix}$ and the zeros are

$$\begin{aligned}
\mathfrak{Z}\left(\begin{bmatrix} \lambda F + G \\ C \end{bmatrix}\right) &= \bigcup_{k=1}^{s} \mathfrak{Z}\left(\begin{bmatrix} \mathcal{L}_{\epsilon_k} \\ C_k \end{bmatrix}\right) \cup \mathfrak{Z}(\mathcal{J}) \cup \mathfrak{Z}(\mathcal{N}) \cup \mathfrak{Z}(\mathcal{M}) \\
&= \bigcup_{k=1}^{s} \left\{\lambda_1^{(k)}, \ldots, \lambda_{\epsilon_k}^{(k)}\right\} \cup \mathfrak{Z}(\mathcal{J}) \\
&= \{\lambda_1, \ldots, \lambda_\epsilon\} \cup \mathfrak{Z}(\lambda F + G) \subset \mathbb{C}_-,
\end{aligned}$$

where we used that only the blocks of type $\mathcal{J}$ in the Kronecker canonical form contain zeros. By Lemma 3.3 this proves that C is a stabilizing controller. □

Now consider the special case of

$$\lambda F + G := \lambda \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} -A & -B \end{bmatrix} \in \mathbb{C}[\lambda]_1^{n,n+m}, \tag{7}$$

where $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,m}$. Then the Kronecker canonical form only contains blocks of type $\mathcal{L}$ and $\mathcal{J}$ (since F has full row rank; Homework, Series 5, Task 8). If we further assume that (A, B) is controllable, then (7) does not have any zeros and thus the Kronecker canonical form cannot have blocks of type $\mathcal{J}$. Thus there are only blocks of type $\mathcal{L}$ and thus in the notation of Theorem 2 we have $\rho = 0$, $\epsilon = n$, and $\epsilon + s = q = n + m$ which implies $s = m$. This means that the controller from Theorem 2 has size m-by-$(n + m)$.

Corollary 3. *Let $(A, B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ be controllable and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then there exist $C_1 \in \mathbb{C}^{m,n}$ and $C_2 \in \mathbb{C}^{m,m}$ such that*

$$\mathfrak{Z}\left(\begin{bmatrix} \lambda I - A & -B \\ C_1 & C_2 \end{bmatrix}\right) = \{\lambda_1, \ldots, \lambda_n\},$$

where C_2 is invertible.

Proof. With the definition from (7) we obtain from Theorem 2 the existence of a regular, stabilizing controller $C \in \mathbb{C}^{m,n+m}$ that satisfies

$$\begin{aligned}
\mathfrak{Z}\left(\begin{bmatrix} \lambda I - A & -B \\ C_1 & C_2 \end{bmatrix}\right) &= \{\lambda_1, \ldots, \lambda_n\} \\
\operatorname{rank}\left(\begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix}\right) &= n + m.
\end{aligned}$$

This means that C_2 has to have full column rank and, since $C_2 \in \mathbb{C}^{m,m}$ is quadratic, C_2 is also invertible. □