First order systems

Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $P = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$ be its Smith form. The problem with the Smith form is that even for first order polynomial matrices

$$P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q},$$

the computed matrices S, D, T can have (arbitrary) high degree.

Example 1. To compute the Smith form of $P \in \mathbb{C}[\lambda]_1^{3,3}$ given by

$$P(\lambda) := \lambda \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 7 & 1 & \\ & 7 & 1 \\ & & 7 \end{bmatrix} = \begin{bmatrix} \lambda - 7 & -1 & \\ & \lambda - 7 & -1 \\ & & \lambda - 7 \end{bmatrix},$$

we perform the elementary unimodular transformations

$$\begin{bmatrix} \lambda - 7 & -1 & \\ & \lambda - 7 & -1 \\ & & \lambda - 7 \end{bmatrix} \overset{ⓐ}{\to} \begin{bmatrix} 0 & -1 & \\ (\lambda-7)^2 & \lambda - 7 & -1 \\ & & \lambda - 7 \end{bmatrix} \overset{ⓑ}{\to} \begin{bmatrix} 0 & -1 & \\ (\lambda-7)^2 & 0 & -1 \\ & & \lambda - 7 \end{bmatrix}$$

$$\overset{ⓒ}{\to} \begin{bmatrix} 0 & -1 & \\ 0 & 0 & -1 \\ (\lambda-7)^3 & & \lambda - 7 \end{bmatrix} \overset{ⓓ}{\to} \begin{bmatrix} 0 & -1 & \\ 0 & 0 & -1 \\ (\lambda-7)^3 & & 0 \end{bmatrix} \overset{ⓔ}{\to} \begin{bmatrix} 0 & 1 & \\ 0 & 0 & 1 \\ (\lambda-7)^3 & & 0 \end{bmatrix} \overset{ⓕ}{\to} \begin{bmatrix} 1 & & \\ & 1 & \\ & & (\lambda-7)^3 \end{bmatrix}.$$

In abstract notation we apply from the left

$$S = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}^{ⓔ} \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & (\lambda-7) & 1 \end{bmatrix}^{ⓓ} \cdot \begin{bmatrix} 1 & & \\ (\lambda-7) & 1 & \\ & & 1 \end{bmatrix}^{ⓑ} = \ldots = \begin{bmatrix} -1 & & \\ -(\lambda-7) & -1 & \\ (\lambda-7)^2 & (\lambda-7) & 1 \end{bmatrix}$$

and from the right

$$T = \begin{bmatrix} 1 & & \\ (\lambda-7) & 1 & \\ & & 1 \end{bmatrix}^{ⓐ} \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ (\lambda-7)^2 & & 1 \end{bmatrix}^{ⓒ} \cdot \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}^{ⓕ} = \ldots = \begin{bmatrix} & & 1 \\ 1 & & (\lambda-7) \\ & 1 & (\lambda-7)^2 \end{bmatrix}$$

to obtain the Smith form

$$SPT = \begin{bmatrix} 1 & & \\ & 1 & \\ & & (\lambda-7)^3 \end{bmatrix} \in \mathbb{C}[\lambda]^{3,3}.$$

Here we see that the matrices S, T, D contain entries with degree bigger than one.

Due to this property one does not simply compute the Smith form of a first order matrix polynomial numerically (at least no way is known to the author). A first step towards numerical computations is given by the Kronecker canonical form. In the Kronecker canonical form we only allow pre- and post-multiplications with constant invertible matrices $S \in \mathbb{C}^{p,p}$ and $T \in \mathbb{C}^{q,q}$. Both S and T can have huge condition numbers. For the robustness of a numerical algorithm, however, it would be best to only allow pre- and post-multiplications with unitary matrices.

Theorem 2 (Kronecker canonical form)**.** *Let $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$. Then there exist nonsingular matrices $S \in \mathbb{C}^{p,p}$ and $T \in \mathbb{C}^{q,q}$ and $\epsilon, \rho, \sigma, \eta, s, u, v, w \in \mathbb{N}_0$ such that*

$$\lambda F + G = S \cdot \operatorname{diag}(\mathcal{L}, \mathcal{J}, \mathcal{N}, \mathcal{M}) \cdot T, \tag{KF}$$

where $\mathcal{L} \in \mathbb{C}[\lambda]_1^{\epsilon,\epsilon+s}$, $\mathcal{J} \in \mathbb{C}[\lambda]_1^{\rho,\rho}$, $\mathcal{N} \in \mathbb{C}[\lambda]_1^{\sigma,\sigma}$, and $\mathcal{M} \in \mathbb{C}[\lambda]_1^{\eta+w,\eta}$ can be further partitioned into

$$\begin{aligned} \mathcal{L} &=: \operatorname{diag}(\mathcal{L}_{\epsilon_1}, \dots, \mathcal{L}_{\epsilon_s}), & \mathcal{J} &=: \operatorname{diag}(\mathcal{J}_{\rho_1}, \dots, \mathcal{J}_{\rho_u}), \\ \mathcal{N} &=: \operatorname{diag}(\mathcal{N}_{\sigma_1}, \dots, \mathcal{N}_{\sigma_v}), & \mathcal{M} &=: \operatorname{diag}(\mathcal{M}_{\eta_1}, \dots, \mathcal{M}_{\eta_w}), \end{aligned}$$

with $\epsilon = \epsilon_1 + \ldots + \epsilon_s$, $\rho = \rho_1 + \ldots + \rho_u$, $\sigma = \sigma_1 + \ldots + \sigma_v$, and $\eta = \eta_1 + \ldots + \eta_w$ and the blocks $\mathcal{L}_{\epsilon_j}$, $\mathcal{J}_{\rho_j}$, $\mathcal{N}_{\sigma_j}$, and $\mathcal{M}_{\eta_j}$ have the following form:

1. *Every entry $\mathcal{L}_{\epsilon_j}$ has the size $\epsilon_j \times (\epsilon_j + 1)$, $\epsilon_j \in \mathbb{N}_0$ and the form*

$$\mathcal{L}_{\epsilon_j}(\lambda) := \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}. \tag{1}$$

2. *Every entry $\mathcal{J}_{\rho_j}$ has the size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$ and the form*

$$\mathcal{J}_{\rho_j}(\lambda) := \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \tag{2}$$

where $\lambda_j \in \mathbb{C}$ is a zero of $\lambda F + G$.

3. *Every entry $\mathcal{N}_{\sigma_j}$ has the size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$ and the form*

$$\mathcal{N}_{\sigma_j}(\lambda) := \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{3}$$

4. *Every entry $\mathcal{M}_{\eta_j}$ has the size $(\eta_j + 1) \times \eta_j$, $\eta_j \in \mathbb{N}_0$ and the form*

$$\mathcal{M}_{\eta_j}(\lambda) := \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}. \tag{4}$$

Proof. The very complex proof can be found in [Gan59, p. 37]. □

Definition 3. For $P \in \mathbb{C}[\lambda]_K^{p,q}$ in the form $P(\lambda) = \sum_{i=0}^{K} \lambda^i P_i$ with $P_i \in \mathbb{C}^{p,q}$ we call

$$\lambda F + G := \lambda \begin{bmatrix} I_q & & & \\ & \ddots & & \\ & & I_q & \\ & & & P_K \end{bmatrix} + \begin{bmatrix} 0 & -I_q & & \\ & \ddots & \ddots & \\ & & 0 & -I_q \\ P_0 & \dots & P_{K-2} & P_{K-1} \end{bmatrix} \in \mathbb{C}[\lambda]_1^{p+q(K-1),qK} \tag{5}$$

the *canonical linearization of* P. Furthermore, for $q, r \in \mathbb{N}$ we denote by

$$\Delta_r^q(\lambda) := \begin{bmatrix} I_q \\ \lambda I_q \\ \vdots \\ \lambda^r I \end{bmatrix} \in \mathbb{C}[\lambda]^{(r+1)q,q}$$

and with this for $z \in \mathcal{C}_\infty^q$ we use the notation

$$\Delta_r z := \Delta_r^q\left(\tfrac{d}{dt}\right) z = \begin{bmatrix} z \\ z^{(1)} \\ \vdots \\ z^{(r)} \end{bmatrix} \in \mathcal{C}_\infty^{(r+1)q}.$$

In the following Lemma we show that the system given by the canonical linearization $\mathfrak{B}(\lambda F + G)$ contains all the relevant information about the original system $\mathfrak{B}(P)$.

Lemma 4. *Let $\lambda F + G \in \mathbb{C}[\lambda]_1^{p+q(K-1),qK}$ be the canonical linearization of $P \in \mathbb{C}[\lambda]_K^{p,q}$. Then we have the following:*

1. $\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) = q(K-1) + \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$
2. $\operatorname{rank}(\lambda_0 F + G) = q(K-1) + \operatorname{rank}(P(\lambda_0))$ *for all* $\lambda_0 \in \mathbb{C}$
3. $\mathfrak{Z}(\lambda F + G) = \mathfrak{Z}(P)$.
4. $\mathfrak{B}(\lambda F + G) = \Delta_{K-1}^q\left(\tfrac{d}{dt}\right)\mathfrak{B}(P)$

Proof. Let P have the form $P(\lambda) = \sum_{i=0}^K \lambda^i P_i$. Then introduce the notation

$$\begin{array}{rcl} P^{\langle 0\rangle}(\lambda) & := & P_K \\ P^{\langle 1\rangle}(\lambda) & := & \lambda P_K + P_{K-1} \\ & \vdots & \\ P^{\langle j\rangle}(\lambda) & := & \displaystyle\sum_{i=0}^{j} \lambda^i P_{K-j+i} = \sum_{i=K-j}^{K} \lambda^{i-K+j} P_i \qquad \text{for } j = 0, \dots, K, \end{array}$$

such that $P^{\langle K\rangle}(\lambda) = P(\lambda)$ and perform the ("block elementary") unimodular transformations

$$\lambda F + G = \begin{bmatrix} \lambda I & -I & & & \\ & \ddots & \ddots & & \\ & & \lambda I & -I & \\ & & & \lambda I & -I \\ P_0 & \dots & P_{K-3} & P_{K-2} & P^{\langle 1\rangle}(\lambda) \end{bmatrix} \overset{\text{(a2)}}{\to} \begin{bmatrix} \lambda I & -I & & & \\ & \ddots & \ddots & & \\ & & \lambda I & -I & \\ & & & \lambda I & -I \\ P_0 & \dots & P_{K-3} & P^{\langle 2\rangle}(\lambda) & 0 \end{bmatrix}$$

$$\overset{\text{(b2)}}{\to} \begin{bmatrix} \lambda I & -I & & & \\ & \ddots & \ddots & & \\ & & \lambda I & -I & \\ & & & 0 & -I \\ P_0 & \dots & P_{K-3} & P^{\langle 2\rangle}(\lambda) & 0 \end{bmatrix} \overset{\text{(a3)}}{\to} \begin{bmatrix} \lambda I & -I & & & \\ & \ddots & \ddots & & \\ & & \lambda I & -I & \\ & & & 0 & -I \\ P_0 & \dots & P^{\langle 3\rangle}(\lambda) & 0 & 0 \end{bmatrix}$$

$$\overset{\text{(b3)}}{\to} \begin{bmatrix} \lambda I & -I & & & \\ & \ddots & \ddots & & \\ & & 0 & -I & \\ & & & 0 & -I \\ P_0 & \dots & P^{\langle 3\rangle}(\lambda) & 0 & 0 \end{bmatrix} \to \dots \overset{\text{(aK)}}{\to} \begin{bmatrix} \lambda I & -I & & & \\ & \ddots & \ddots & & \\ & & 0 & -I & \\ & & & 0 & -I \\ P^{\langle K\rangle}(\lambda) & & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{\text{(bK)}}{\to} \begin{bmatrix} 0 & -I & & & \\ & \ddots & \ddots & & \\ & & 0 & -I & \\ & & & 0 & -I \\ P^{\langle K\rangle}(\lambda) & & 0 & 0 & 0 \end{bmatrix}.$$

In abstract notation we apply from the left

$$\begin{aligned} S &= \begin{bmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & & I & \\ P^{\langle K-1\rangle}(\lambda) & & & & I \end{bmatrix}^{\text{(aK)}} \cdot \ldots \cdot \begin{bmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & & I & \\ & & P^{\langle 2\rangle}(\lambda) & & I \end{bmatrix}^{\text{(a3)}} \cdot \begin{bmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & & I & \\ & & & P^{\langle 1\rangle}(\lambda) & I \end{bmatrix}^{\text{(a2)}} \\ &= \begin{bmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & & I & \\ P^{\langle K-1\rangle}(\lambda) & \cdots & P^{\langle 2\rangle}(\lambda) & P^{\langle 1\rangle}(\lambda) & I \end{bmatrix} \end{aligned}$$

and from the right

$$\begin{aligned} T &= \begin{bmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & & I & \\ & & & \lambda I & I \end{bmatrix}^{\text{(b2)}} \cdot \begin{bmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & \lambda I & I & \\ & & & & I \end{bmatrix}^{\text{(b3)}} \cdot \ldots \cdot \begin{bmatrix} I & & & & \\ \lambda I & \ddots & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix}^{\text{(bK)}} \\ &= \begin{bmatrix} I & & & & \\ \lambda I & I & & & \\ \vdots & \vdots & \ddots & & \\ \lambda^{K-2} I & \lambda^{K-3} I & \cdots & I & \\ \lambda^{K-1} I & \lambda^{K-2} I & \cdots & \lambda I & I \end{bmatrix} \end{aligned}$$

to obtain that

$$S(\lambda)\,(\lambda F + G)\,T(\lambda) = \begin{bmatrix} 0 & -I & & \\ & \ddots & \ddots & \\ & & 0 & -I \\ P^{\langle K\rangle}(\lambda) & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & -I & & \\ & \ddots & \ddots & \\ & & 0 & -I \\ P(\lambda) & & & 0 \end{bmatrix}, \tag{6}$$

which implies 1. Since S and T are unimodular there exist constants $c_S, c_T \in \mathbb{C}\backslash\{0\}$ such that $\det S(\lambda_0) = c_S \neq 0$ and $\det T(\lambda_0) = c_T \neq 0$ for all $\lambda_0 \in \mathbb{C}$. Thus the matrices $S(\lambda_0)$ and $T(\lambda_0)$ are invertible (over $\mathbb{C}$) for all $\lambda_0 \in \mathbb{C}$. We conclude that for $\lambda_0 \in \mathbb{C}$ we have by using (6) that

$$\operatorname{rank}(\lambda_0 F + G) \;=\; \operatorname{rank}\left(S(\lambda_0)(\lambda_0 F + G)T(\lambda_0)\right) = q(K-1) + \operatorname{rank}\left(P(\lambda_0)\right),$$

which implies 2. Point 3. then follows by combining 1. and 2. together with Lemma 1.9. Finally, for point 4. we note that

$$\mathfrak{B}(\lambda F + G) \;=\; T\left(\tfrac{d}{dt}\right)\mathfrak{B}(S(\lambda)(\lambda F + G)T(\lambda)) = T\left(\tfrac{d}{dt}\right)\mathfrak{B}\left(\begin{bmatrix} 0 & -I & & \\ & \ddots & \ddots & \\ & & 0 & -I \\ P(\lambda) & & & 0 \end{bmatrix}\right)$$

$$
\begin{aligned}
&= \left[\Delta^q_{K-1}\left(\tfrac{d}{dt}\right) \quad \star \quad \cdots \quad \star\right] \left\{(z,w) \in \mathcal{C}^q_\infty \times \mathcal{C}^{q(K-1)}_\infty \;\middle|\; \begin{bmatrix} 0 & -I_{q(K-1)} \\ P\left(\tfrac{d}{dt}\right) & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = 0 \right\} \\
&= \left[\Delta^q_{K-1}\left(\tfrac{d}{dt}\right) \quad \star \quad \cdots \quad \star\right] \left\{(z,w) \in \mathcal{C}^q_\infty \times \mathcal{C}^{q(K-1)}_\infty \;\middle|\; P\left(\tfrac{d}{dt}\right) z = 0 \text{ and } w = 0 \right\} \\
&= \Delta^q_{K-1}\left(\tfrac{d}{dt}\right) \mathfrak{B}(P),
\end{aligned}
$$

which finishes the proof. □

In particular point 4. shows that the first q elements of the system $\mathfrak{B}(\lambda F + G)$ give the original behavior $\mathfrak{B}(P)$. The other elements are derivatives of the trajectories of $\mathfrak{B}(P)$ and can be considered latent variables. In other words, if $\lambda F + G$ is the canonical linearization of P then

$$
\mathfrak{B}(P) = \left\{ z \in \mathcal{C}^q_\infty \;\middle|\; \exists \ell \in \mathcal{C}^{q(K-1)}_\infty \text{ such that with } y := \begin{bmatrix} z \\ \ell \end{bmatrix} \text{ we have } F\dot{y} + Gy = 0 \right\},
$$

is a latent variable description of $\mathfrak{B}(P)$. This latent variable description has the advantage, that it only involves a derivative of first order and thus, one can use the Kronecker canonical form.

Lemma 5. *Let the Kronecker form of $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}_1$ be given by (KF). Then the (compact) behavior is given by*

$$
\begin{aligned}
\mathfrak{B}(\lambda F + G) &= T^{-1} \left\{ \begin{bmatrix} \Delta_{\epsilon_1} z_1 \\ \vdots \\ \Delta_{\epsilon_s} z_s \\ e^{\mathcal{J}(0)t} \hat{x} \\ 0_{\sigma+\eta} \end{bmatrix} \;\middle|\; z_1, \ldots, z_s \in \mathcal{C}^1_\infty, \hat{x} \in \mathbb{C}^\rho \right\}, \\
\mathfrak{B}_c(\lambda F + G) &= T^{-1} \left\{ \begin{bmatrix} \Delta_{\epsilon_1} z_1 \\ \vdots \\ \Delta_{\epsilon_s} z_s \\ 0_{\rho+\sigma+\eta} \end{bmatrix} \;\middle|\; z_1, \ldots, z_s \in \mathcal{C}^1_c \right\},
\end{aligned}
$$

Proof. Look at the behavior of each block in the Kronecker canonical form separately. Then assemble the obtained behaviors. The complete proof is Homework (Series 3, Task 1). □

References

[Gan59] F.R. Gantmacher. *The Theory of Matrices II.* Chelsea Publishing Company, New York, NY, 1959.