Motivation

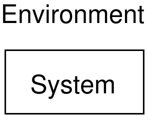
Systems and control theory
SS 2012
TU Berlin

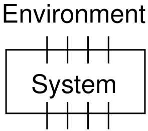
Definition

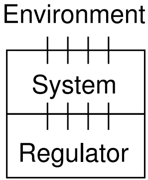
A *system* is an entity which can be separated from its environment.

Systems theory tries to describe the interaction of a system with its environment

Control theory tries to influence a system, so that it has favorable properties.







Steam engine

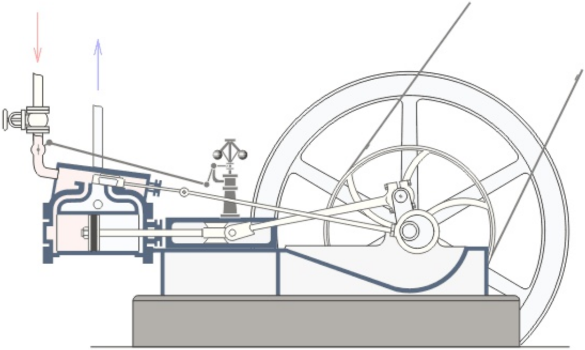
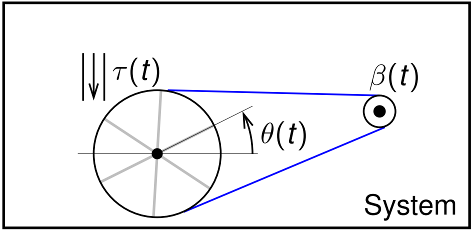


Image from Wikipedia
Drawn by Panther

`http://en.wikipedia.org/wiki/Steam_engine`
`http://commons.wikimedia.org/wiki/User:Panther`

The system



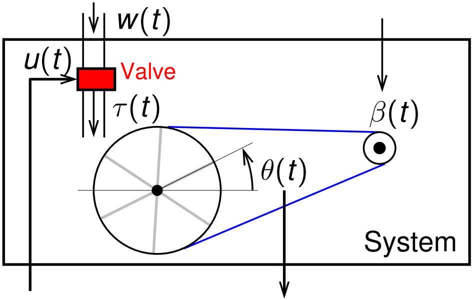
A simplistic model of the stream engine is given by

$$j\ddot{\theta}(t) = -\mu\dot{\theta}(t) + k\tau(t) - \beta(t),$$

where

$\theta(t) \in \mathbb{R}$	–	rotation angle	$j \in \mathbb{R}_{>0}$	–	moment of inertia
$\tau(t) \in \mathbb{R}$	–	applied torque	$\mu \in \mathbb{R}_{\geq 0}$	–	friction coefficient
$\beta(t) \in \mathbb{R}$	–	applied load	$k \in \mathbb{R}_{>0}$	–	gain coefficient

Connections to the environment



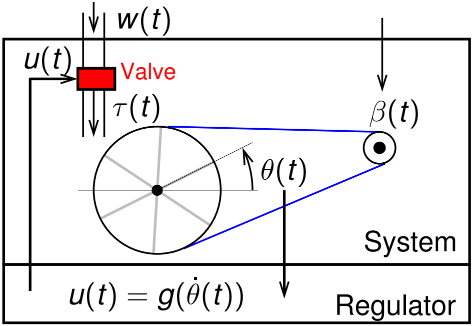
The torque $\tau(t)$ is determined by the steam pressure $w(t) \in \mathbb{R}$ and the position of the valve $u(t) \in [0, 1]$, e.g., we could have

$$\tau(t) = f(w(t), u(t)) := u(t)w(t).$$

One could say that the variables, by which the system interacts with its environment are given by

$$w, \beta, u, \text{ and } \theta.$$

The regulator



Historically, humans were interested in a constant speed $\dot{\theta}(t) \approx v_0$ of the engine, subject to (reasonable) fluctuations in the stream pressure $w(t)$ and load $\beta(t)$.

Idea: Let the valve position $u(t)$ be determined by the speed of the engine.

The flyball governor

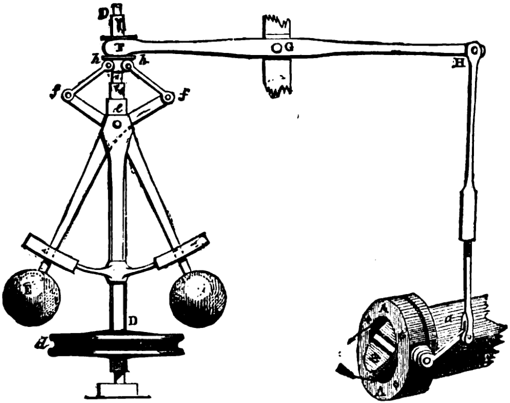


FIG. 4.—*Governor and Throttle-Valve.*

Image from Wikipedia
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`http://en.wikipedia.org/wiki/Centrifugal_governor`

Boulton and Watt governor

Also search Google and YouTube for *flyball governor*.

Image from Wikipedia http://en.wikipedia.org/wiki/Centrifugal_governor
Foto by Dr. Mirko Junge http://commons.wikimedia.org/wiki/User:DrJunge

Steam engine

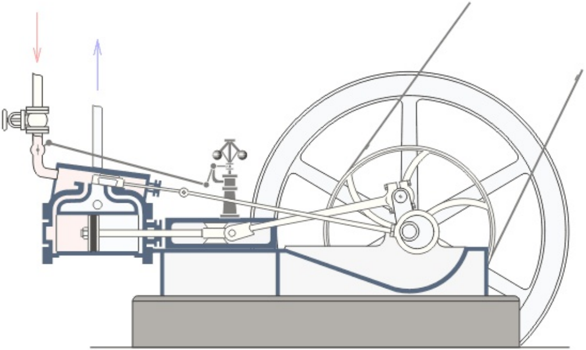


Image from Wikipedia
Drawn by Panther

http://en.wikipedia.org/wiki/Steam_engine
http://commons.wikimedia.org/wiki/User:Panther

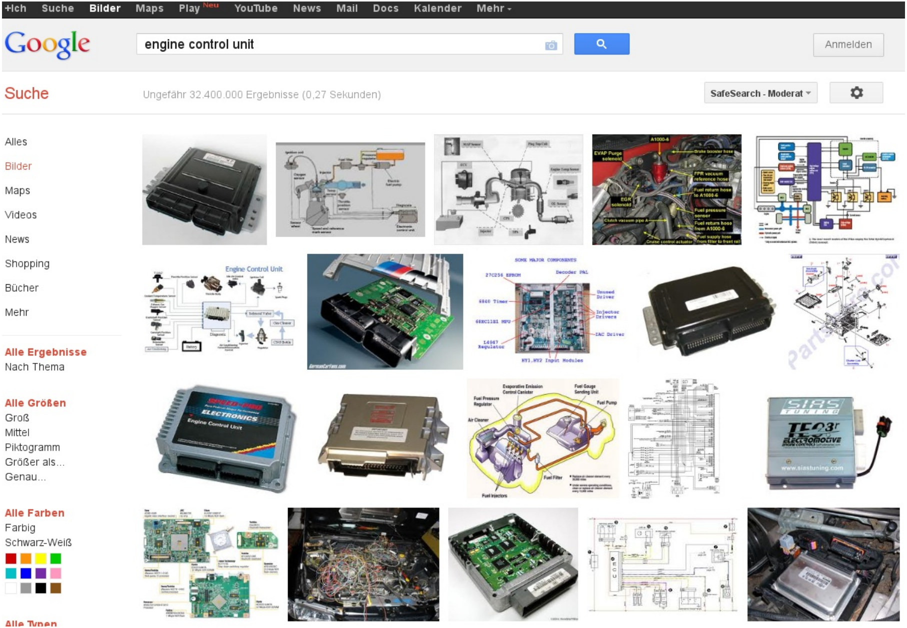
Questions

1. How can the flyball governor be modeled?
2. How can the flyball governor be tuned, so that the steam engine indeed runs at a constant speed?
3. How can unwanted oscillations be avoided?
4. Is the model for the steam engine appropriate to describe reality?

cf. ”On governors”, James Clerk Maxwell, 1868

These questions will (mostly) not be considered/answered in this course.

Modern “flyball governors”



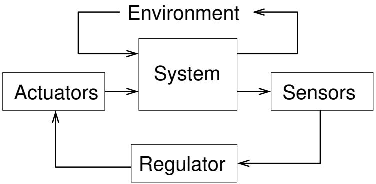
Different approaches

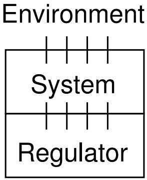
Input-Output systems

- Anthropomorphic viewpoint
- appropriate for electric control units

Behavioral systems

- simpler
- in some cases more appropriate (see electric circuits below)





Closed-loop vs. Open-loop

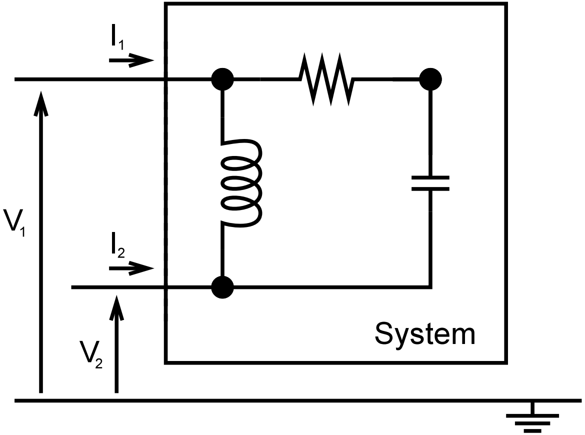
For input-output systems, one distinguishes two kinds of control:

- **open-loop** (german: ”Steuerung”) one fixes a control law a-priori (only Actuators necessary)
- **closed-loop** (german: ”Regelung”, also: feedback control) choose the control, based on simultaneous measurements (Actuators and Sensors necessary).

Open-loop control can not handle unforseen disturbances (i.e., it is not so *robust*).

Here we only consider closed-loop control.

A behavioral example

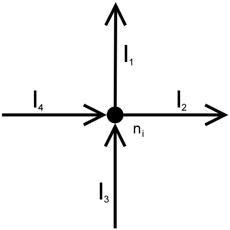


What can be observed from the outside are:

I_1, I_2 the currents going into the circuit

V_1, V_2 the potential differences against the ground (i.e. voltages)

Kirchhoff's current law ...

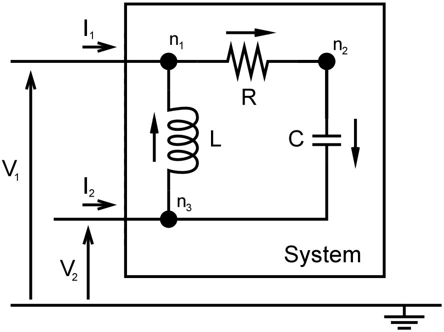


... states that a node n_i cannot generate charge, i.e., that the currents that go into one node sum up to zero:

$$I_1 + I_2 - I_3 - I_4 = 0.$$

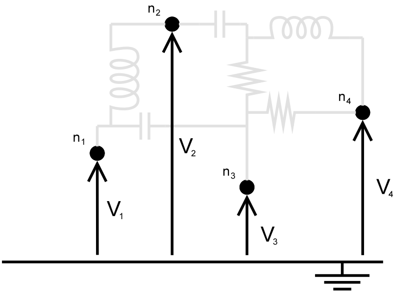
Kirchhoff's current law in example

Let I_L, I_R, I_C denote the currents through the named components.



$$
\begin{aligned}
n_1: &\quad 0 = I_1 - I_R + I_L \\
n_2: &\quad 0 = I_R - I_C \\
n_3: &\quad 0 = I_2 - I_L + I_C
\end{aligned}
$$

Kirchhoff's voltage law ...



... states that the voltages in a cycle up to zero:

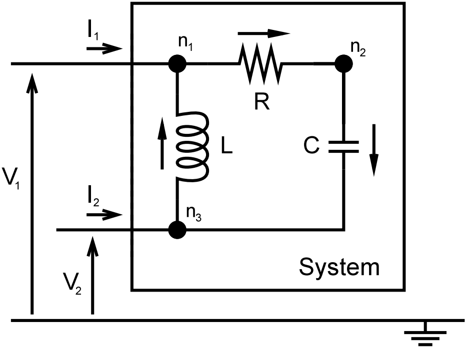
$$V\Big|_{n_1}^{n_2} + V\Big|_{n_2}^{n_3} + V\Big|_{n_3}^{n_4} + V\Big|_{n_4}^{n_1} = 0,$$

where $V\Big|_{n_i}^{n_j}$ denotes the voltage from node n_i to node n_j.

In the graphic above we have (with $V_i := V\Big|_{\text{ground}}^{n_i}$):

$$\begin{aligned} 0 &= V\Big|_{n_1}^{n_2} + V\Big|_{n_2}^{\text{ground}} + V\Big|_{\text{ground}}^{n_1} = V\Big|_{n_1}^{n_2} - V_2 + V_1, \\ \Rightarrow \qquad V\Big|_{n_1}^{n_2} &= V_2 - V_1. \end{aligned}$$

Kirchhoff's voltage law in example

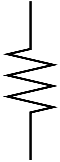


By abusing the notation $n_i := V\big|_{\text{ground}}^{n_i}$ we can state

$$\begin{aligned} V_1 &= n_1, \\ V_3 &= n_3. \end{aligned}$$

The RLC-components

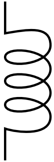
A resistor
with resistance R



is described by

$$V(t) = RI(t)$$

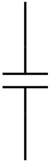
An inductor
with inductance L



is described by

$$L\dot{I}(t) = V(t)$$

A capacitor
with capacitance C



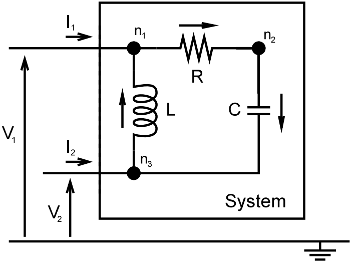
is described by

$$I(t) = C\dot{V}(t)$$

where V is the voltage across the component and
I is the current through the component.

The RLC-components in example

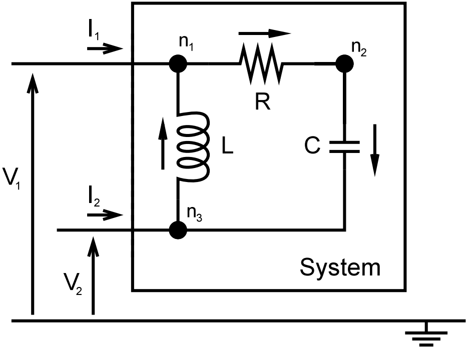
As before, let n_1, n_2, n_3 denote the voltages of the nodes against the ground. The voltages across the components can then be obtained by Kirchhoff's voltage law.



We obtain the equations:

$$\begin{aligned} L\frac{d}{dt} I_L(t) &= (n_1(t) - n_3(t)), \\ R I_R(t) &= (n_2(t) - n_1(t)), \\ C\frac{d}{dt}(n_3(t) - n_2(t)) &= I_C(t). \end{aligned}$$

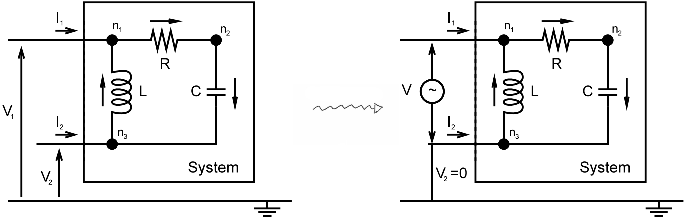
The complete equations



Writing all equations together we get

$$\begin{bmatrix} 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ I_L \\ I_R \\ I_C \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ I_L \\ I_R \\ I_C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V_1 \\ V_2 \end{bmatrix}.$$

The voltage could be the input!

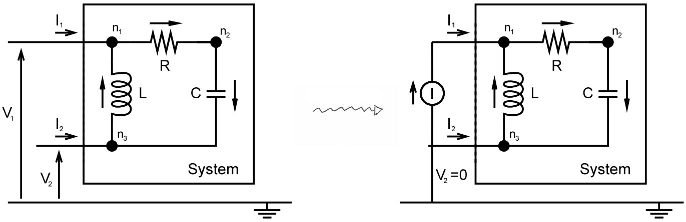


One can solder one wire to the ground $V_2 = 0$ and connect the two wires via a voltage source $V(t) = V_1(t) - V_2(t) = V_1(t)$ and then consider

$$\begin{bmatrix} 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ I_L \\ I_R \\ I_C \\ I_1 \\ I_2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ I_L \\ I_R \\ I_C \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} V(t).$$

One can show that for every sufficiently smooth V and consistent initial conditions this system has a unique solution.

The current could be the input!



One can solder one wire to the ground $V_2 = 0$ and connect the two wires via a current source $I(t) = I_1(t) = -I_2(t)$ and then consider

$$\begin{bmatrix} 0 & 0 & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{\cdot}{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ I_L \\ I_R \\ I_C \\ V_1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ I_L \\ I_R \\ I_C \\ V_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} I(t).$$

One can show that for every sufficiently smooth I and consistent initial conditions this system has a unique solution. There is a redudancy among the equations 4-6.

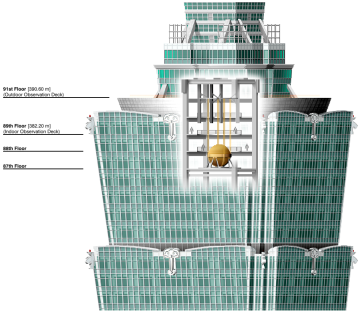
Conclusion

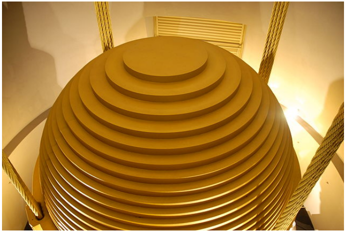
- The equations for RLC-circuits describe how voltages and currents interact. In general, neither the voltages nor the currents are inputs or outputs.
- Every electric RLC-circuit with m wires sticking out, n_1 internal nodes, and n_2 components is described by a system of the form

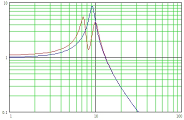
$$\begin{bmatrix} E_1 & E_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{n} \\ I_c \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & C_1 \\ D_1 & 0 \end{bmatrix} \begin{bmatrix} n \\ I_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C_2 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix},$$

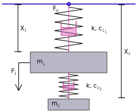
where $n(t) \in \mathbb{R}^{n_1}$, $I_c(t) \in \mathbb{R}^{n_2}$, $E_1, A_1 \in \mathbb{R}^{n_2,n_1}$, $E_2, A_2 \in \mathbb{R}^{n_2,n_2}$, $C_1 \in \mathbb{R}^{n_1,n_2}$, $C_2 \in \mathbb{R}^{n_1,m}$, $D_1 \in \mathbb{R}^{m,n_1}$, $D_2 \in \mathbb{R}^{m,m}$, $I(t), V(t) \in \mathbb{R}^m$.

Tuned mass damper









Images from Wikipedia:
`http://en.wikipedia.org/wiki/Tuned_mass_damper`
by authors *Someformofhuman*, *guillom*, *Greglocock*, and another

Also google for "tuned mass damper" and look at the videos!

The 1D wave equation

The 1D wave equation in the unknown $y : \mathbb{R} \times [0,1] \to \mathbb{R}$ is

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} y(t,q) &= \frac{\partial^2}{\partial y^2} y(t,q) + \begin{bmatrix} b_1(q) & \cdots & b_m(q) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} + w(q,t), \\
0 &= y(t,0) = y(t,1), \\
0 &= y(0,q) = \dot{y}(0,q),
\end{aligned}$$

where $q \in [0,1]$, $t \in [0,\infty)$ and

- $y(t,q)$ describes the state of the wave at time t and position $q \in [0,1]$,
- $b_1, \ldots, b_m : [0,1] \to \mathbb{R}$ are arbitrary fixed functions which are defined by the problem,
- $u_1, \ldots, u_m : \mathbb{R} \times \mathbb{R}$ are the controls (by which we can influence the system), and
- $w : \mathbb{R} \times [0,1] \to \mathbb{R}$ is an external perturbation.

Spacial discretization

To approximate y on an equidistant grid with $n \in \mathbb{N}$ intervals

$$x_i(t) := y\left(t, \frac{i}{n}\right), \qquad \text{for } \ i = 0, \ldots, n,$$

using the boundary conditions, and the finite difference equation

$$\begin{aligned}
\ddot{x}_i(t) \quad &= \quad \frac{\partial^2}{\partial t^2} y\left(t, \tfrac{i}{n}\right) = \frac{\partial^2}{\partial y^2} y\left(t, \tfrac{i}{n}\right) + \begin{bmatrix} b_1\left(\tfrac{i}{n}\right) & \cdots & b_m\left(\tfrac{i}{n}\right) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} + w\left(\tfrac{i}{n}, t\right) \\
&\approx \quad \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{\frac{1}{n^2}} + \begin{bmatrix} b_1\left(\tfrac{i}{n}\right) & \cdots & b_m\left(\tfrac{i}{n}\right) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} + w\left(\tfrac{i}{n}, t\right), \qquad \text{for } i = 1, \ldots n-1
\end{aligned}$$

leads to the approximate system

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-2}(t) \\ x_{n-1}(t) \end{bmatrix} = n^2 \underbrace{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}}_{=:A \in \mathbb{R}^{n-1,n-1}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-2}(t) \\ x_{n-1}(t) \end{bmatrix}}_{=:x \in \mathbb{R}^{n-1}} + \underbrace{\begin{bmatrix} b_1\left(\frac{1}{n}\right) & \cdots & b_m\left(\frac{1}{n}\right) \\ b_1\left(\frac{2}{n}\right) & \cdots & b_m\left(\frac{2}{n}\right) \\ \vdots & & \vdots \\ b_1\left(\frac{n-2}{n}\right) & \cdots & b_m\left(\frac{n-2}{n}\right) \\ b_1\left(\frac{n-1}{n}\right) & \cdots & b_m\left(\frac{n-1}{n}\right) \end{bmatrix}}_{=:B \in \mathbb{R}^{n-1,m}} u(t) + \underbrace{\begin{bmatrix} w\left(t, \frac{1}{n}\right) \\ w\left(t, \frac{2}{n}\right) \\ \vdots \\ w\left(t, \frac{n-2}{n}\right) \\ w\left(t, \frac{n-1}{n}\right) \end{bmatrix}}_{=:v(t) \in \mathbb{R}^{n-1}}.$$

The resulting system

Let denote

$$\mathcal{C}_\infty^q := \left\{ x : \mathbb{R} \to \mathbb{R}^q \middle| x \text{ is infinitely often differentiable} \right\},$$

and (for ease of notation) replace $n-1$ by n.

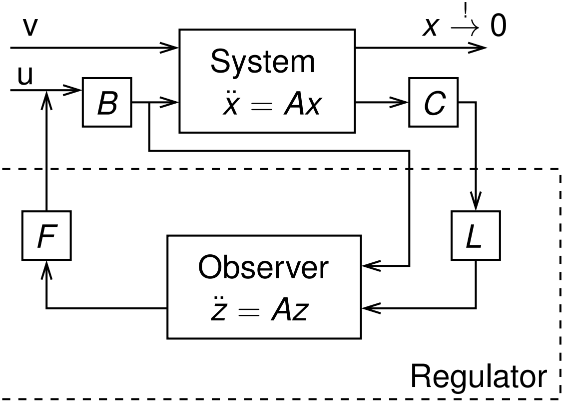
Then the 1D-Wave equation on $[0, 1]$ is approximated by the linear system

$$\ddot{x}(t) = Ax(t) + Bu(t) + v(t).$$

with state $x : \mathcal{C}_\infty^n$, $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, input $u \in \mathbb{C}^m$, and external perturbation $v \in \mathcal{C}_\infty^n$.

The wave equation in pictures

Let $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{\ell,n}$, $F \in \mathbb{R}^{m,n}$, $L \in \mathbb{R}^{n,\ell}$.



We call $x \in \mathcal{C}_\infty^n$ the state, $u \in \mathcal{C}_\infty^m$ the input, $y = Cx$ the output, $v \in \mathcal{C}_\infty^n$ the external perturbation.

$\rightarrow$ `RUNME.m`

Automatic control of guided missiles

“Modern Homing Missile Guidance Theory and Techniques” by Neil F. Palumbo, Ross A. Blauwkamp, and Justin M. Lloyd, Johns Hopkins APL Technical Digest, Volume 29, Number 1 (2010)

From the abstract:
“Classically derived homing guidance laws, such as proportional navigation, can be highly effective when the homing missile has significantly more maneuver capability than the threat. As threats become more capable, however, higher performance is required from the missile guidance law to achieve intercept. To address this challenge, most modern guidance laws are derived using linear-quadratic optimal control theory to obtain analytic feedback solutions.”

See also: `http://techdigest.jhuapl.edu/TD/td2901/index.htm`

Stochastic control and delay

Stochastic version of what we will seem to play role in many practical applications, e.g., engineering problems, biological systems, traffic networks, communication networks, finance, ...

In practice it takes time to send the data from the sensors to regulator and back to the actuators. One can try to develop theory to handle this *delay*.

In this course *stochastic control* and *delays* will not be covered.

Further applications

Control engineering plays an important role in

1. automotive engineering
2. aerospace engineering
3. electrical engineering
4. chemical engineering
5. mechanical engineering

Rules

1. Everyone, who attended more than 30 % of the lectures can take the exam in the end.
2. Relevant for the exam is everything, unless explicitly states otherwise.
3. Do the homeworks.
4. On Monday you are required to print all the material you need for the whole week (if any).
5. Mathematical issues can be discussed at any time, at “any” sound level. Non-mathematical discussions are not allowed.

Remark

1. I want to teach you all I know about systems and control theory! This requires you (and me) to work hard.
2. If it does not feel right, do not work to long an exercise. May there is a mistake in it or you misunderstood something simple.
3. If you are not learning, do something against it, e.g., quit.
4. The script is not in a finished state. It is subject to change, and the quality of the files that will be uploaded on the website might be a bit messy.
5. This lecture/talk was not representative for the rest of the semester. We will work a lot on the blackboard.
6. Your remarks are welcome.