Systems and Control Theory

Chapter 1: Introduction

In the most general form a system is a subset $\mathfrak{B}$ (also called behavior or system) of some superset $\mathcal{U}$ (also called universe)

Example:

To discribe the state of matter (Aggregatzustand) of water, one could choose the universe

$\mathcal{U} := \{solid, liquid, gas\} \times [-273,15, \infty)$.

Then the behavior is often chosen to be

$$\mathfrak{B} := (\{Solid\} \times [-273,15, 0)) \cup (\{liquid\} \times [0, 100]) \cup (\{gas\} \times [100, \infty))$$

① behavior is not a map!

A (continuous-time) dynamical system is one, where the universe takes the form

$\mathcal{U} := \{z : \underset{\text{time}}{\mathbb{R}} \to W\}$ for some arbitray set W, called the signal space. Then the elements of the behavior $z \in \mathfrak{B} \subseteq \mathcal{U}$ are called the trajectories of $\mathfrak{B}$.

The evaluation $z(t)$ for $t \in \mathbb{R}$ is called the value of z at time t.

Discrete-time dynamical systems are systems, where the universe takes the form $\mathcal{U} := \{z : \mathbb{Z} \to W\}$.

A dynamical system is called linear, if W is a linear space and $\mathfrak{B} \subseteq \mathcal{U}$ is a linear subspace (with addition and scalar multiplication defined in the canonical way)

A dynamical system is called time-invariant, if $z(\cdot) \in \mathfrak{L}e \Rightarrow z(\cdot + \tau) \in \mathfrak{L}e$ for all $\tau \in \mathbb{R}$.

The Kernel and image representation

Let $\mathbb{C}[\lambda]$ denote the polynomials with coefficients in $\mathbb{C}$ and $\mathbb{C}[\lambda]_K$ denote those elements of $\mathbb{C}[\lambda]$ which have degree $\leq K$.

$\mathbb{C}[\lambda]^{p,q}$ and $\mathbb{C}[\lambda]_K^{p,q}$ denote the p-by-q matrices with entries in $\mathbb{C}[\lambda]$ and $\mathbb{C}[\lambda]_K$ and can be written as

$$(*) \quad P(\lambda) = \sum_{i=0}^{K} \lambda^i P_i = \lambda^K P_K + \ldots + \lambda P_1 + P_0 = \begin{bmatrix} P_{11}(\lambda) & \ldots & P_{1q}(\lambda) \\ \vdots & & \vdots \\ P_{p1}(\lambda) & \ldots & P_{pq}(\lambda) \end{bmatrix}$$

where $P_0, \ldots, P_K \in \mathbb{C}^{p,q}$ are matrices in the usual sense.

Define $\mathcal{C}_\infty^q := \{ z : \mathbb{R} \to \mathbb{C}^q \mid z \text{ infinitely often differentiable} \}$

Let $P \in \mathbb{C}[\lambda]^{p,q}$ be given in the form $(*)$ and let $z \in \mathcal{C}_\infty^q$. Then (in a slightly symbolic fashion) we define $P(\frac{\partial}{\partial t}) z \in \mathcal{C}_\infty^p$ through

$$\left(P\left(\tfrac{\partial}{\partial t}\right) z\right)(t) = \left(\left(\tfrac{\partial}{\partial t}\right)^K P_K + \ldots + \left(\tfrac{\partial}{\partial t}\right) P_1 + P_0\right) z(t)$$
$$:= P_k z^{(k)}(t) + \ldots + P_1 z'(t) + P_0 z(t),$$

where $z^{(i)}$ denotes the i-the derivative of z.

Definition 1.1.

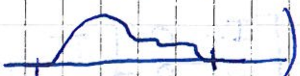
Let $P \in \mathbb{C}[\lambda]^{p,q}$, $U \in \mathbb{C}[\lambda]^{q,m}$. Then we call

$$\mathrm{Kernel}_{\mathcal{C}_\infty}\left(P\left(\tfrac{\partial}{\partial t}\right)\right) := \left\{ z \in \mathcal{C}_\infty^q \mid P\left(\tfrac{\partial}{\partial t}\right) z(t) = 0 \right\}$$

the (behavioral) system in Kernel representation given by P. Similar, we call

$$\mathrm{Image}_{\mathcal{C}_\infty}\left(U\left(\tfrac{\partial}{\partial t}\right)\right) := \left\{ U\left(\tfrac{\partial}{\partial t}\right) \alpha \mid \alpha \in \mathcal{C}_\infty^m \right\}$$

the (behavioral) system in image representation given by U.

Let $\mathcal{C}^q_c$ denote those elements of $\mathcal{C}^q_\infty$ which have compact support. () Then

kernel$_{\mathcal{C}_c}(P(\frac{\partial}{\partial t}))$ and image$_{\mathcal{C}_c}(M(\frac{\partial}{\partial t}))$ are

defined analoguously. We use the shorthands

$\mathfrak{B}(P) := \text{Kernel}_{\mathcal{C}_\infty}(P(\frac{\partial}{\partial t}))$ and $\mathfrak{B}_c(P) = \text{Kernel}_{\mathcal{C}_c}(P(\frac{\partial}{\partial t}))$

and call them the <u>(compact) behavior of P.</u>

<u>Homework:</u> Every kernel and image representation defines a linear time-invariant dynamical system.

<u>Rational functions</u>

Let $\mathbb{C}(\lambda)$ denote the set of rational functions with coefficients in $\mathbb{C}$. Obviously $\mathbb{C}[\lambda] \subseteq \mathbb{C}(\lambda)$. (e.g.: $\frac{3\lambda^2+1}{\lambda-1} \in \mathbb{C}(\lambda)$) The set $\mathbb{C}(\lambda)$ is a field (Körper). $\mathbb{C}[\lambda]$ is only a ring. Denoting the p-by-q matrices with entries from $\mathbb{C}(\lambda)$ by $\mathbb{C}(\lambda)^{p,q}$ (also called <u>rational matrices</u>) one can apply many of the results from linear algebra to the elements of $\mathbb{C}(\lambda)^{p,q}$. For example, one can attribute a unique <u>rank (over $\mathbb{C}(\lambda)$)</u> $\text{rank}_{\mathbb{C}(\lambda)}(R)$ to every $R \in \mathbb{C}(\lambda)^{p,q}$. Since $\mathbb{C}[\lambda]^{p,q} \subseteq \mathbb{C}(\lambda)^{p,q}$ we can also attribute a unique rank to every polynomial matrix $P \in \mathbb{C}[\lambda]^{p,q}$, denoted by $\text{rank}_{\mathbb{C}(\lambda)}(P)$. An element of $\mathbb{C}(\lambda)^{p,p} \ni S$ is called <u>invertible (over $\mathbb{C}(\lambda)$)</u> if its rank is full; in this case there exists a <u>unique</u> inverse $S^{-1} \in \mathbb{C}(\lambda)^{p,p}$. We have $\text{rank}_{\mathbb{C}(\lambda)}(SRT) = \text{rank}_{\mathbb{C}(\lambda)}(R)$ for all invertible (over $\mathbb{C}(\lambda)$) S and T of appropriate dimension.

For every $R \in \mathbb{C}(\lambda)^{p,q}$ there exist invertible S, T such that $R = S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T$, $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(R)$. $(\square)$

Partitioning $S = [S_1, S_2]$, $S_1 \in \mathbb{C}(\lambda)^{p,r}$, $T^{-1} = [T_1, T_2]$, $T_1 \in \mathbb{C}(\lambda)^{q,r}$ we find that

$$\begin{aligned}
\operatorname{kernel}_{\mathbb{C}(\lambda)}(R) &:= \{x \in \mathbb{C}(\lambda)^{q,1} : Rx = 0\} \\
&= \{x \in : S^{-1}Rx = 0\} \\
&= \{T^{-1}Tx : S^{-1}RT^{-1}\underbrace{Tx}_{y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \overset{?}{=} 0\} \\
&= \{T^{-1}y \in \mathbb{C}(\lambda)^q : \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} y = 0\} \\
&= \{[T_1, T_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = 0\} = \operatorname{image}_{\mathbb{C}(\lambda)}(T_2)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{image}_{\mathbb{C}(\lambda)} R &:= \{R\alpha : \alpha \in \mathbb{C}(\lambda)^q\} = \{SS^{-1}RT^{-1}T\alpha : \underbrace{T\alpha}_{=:\beta} \in \mathbb{C}(\lambda)^q\} \\
&= \{S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\beta : \underset{\mathbb{C}(\lambda)^q}{\beta \in}\} = \ldots = \operatorname{image}_{\mathbb{C}(\lambda)}(S_1).
\end{aligned}$$

<u>Definition 1.2.</u>

Let $R \in \mathbb{C}(\lambda)^{p,q}$ and $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(R)$ be it rank (over $\mathbb{C}(\lambda)$). Then $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}(\lambda)^{t,r}$ are called <u>kernel- and co-kernel spanning matrices</u> of R, resp., if they fulfill the following properties:

1) $RU = 0$
2) $\operatorname{rank}_{\mathbb{C}(\lambda)}(RV) = r$
3) $[V, U]$ is invertible (over $\mathbb{C}(\lambda)$)

Using $(\square)$ (and the following construction) we see that with $U := T_2$, $V := T_1$ such kernel- and co-kernel matrices always exist:

Lemma 1.3

For every rational matrix there exists a kernel- and co-kernel spanning matrix.

The Smith Form

Definition 1.4:

A square polynomial matrix $Q \in \mathbb{C}[\lambda]^{p,p}$ is called unimodular if its unique inverse (over $\mathbb{C}(\lambda)$) is polynomial, i.e., if there exists a $\tilde{Q} \in \mathbb{C}[\lambda]^{p,p}$ such that $\tilde{Q} Q = Q \tilde{Q} = I$.

Lemma 1.5.

A polynomial matrix ~~Q~~ $Q \in \mathbb{C}[\lambda]^{p,p}$ is unimodular if and only if its determinant $\det Q \in \mathbb{C}[\lambda]$ is a nonzero constant.

Proof.: Since $1 = \det I = \det(Q Q^{-1}) = \det Q \cdot \det Q^{-1}$ we see that (over $\mathbb{C}(\lambda)$) $\det(Q) = \det(Q^{-1})^{-1}$

If Q is unimodular, then both $\det Q$ and $\det Q^{-1}$ are polynomials and that they have to be a nonzero constant.

If on the other hand, $\det Q$ is a nonzero constant we obtain that the inverse of Q is polynomial by using Cramer's rule: $Q^{-1} = \frac{1}{\det Q} \cdot \operatorname{Adj} Q$ ∎

Theorem 1.6 (Smith form)

Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Then there exists unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ such that

$$(SF) \qquad P = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T \quad \text{with} \quad D = \operatorname{diag}(d_1, \dots, d_r) \in \mathbb{C}[\lambda]^{r,r}$$

where $d_1, \dots, d_r \in \mathbb{C}[\lambda]$ with $d_i \neq 0$ for $i = 1, \dots, r$ and

d_{i+1} divides d_i for $i=1,\dots,r-1$. The d_i are unique.

Proof: See Latex $\boxtimes$

Let the Smith form of $P \in \mathbb{C}[\lambda]^{p,q}$ be given by (SF). Partition $T^{-1} =: [T_1, T_2]$, $T_1 \in \mathbb{C}[\lambda]^{q,r}$, $T_2 \in \mathbb{C}[\lambda]^{q,q-r}$. Then we have $\mathfrak{L}(P) = \mathfrak{L}\left(S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T\right)$

HA
$$\overset{\text{HA}}{=} T^{-1}\left(\frac{\partial}{\partial t}\right) \mathfrak{L}\left(\begin{array}{cc} r & q-r \\ \left[\begin{matrix} D \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 0 \end{matrix}\right] \end{array}\right) = \left[T_1\left(\frac{\partial}{\partial t}\right), T_2\left(\frac{\partial}{\partial t}\right)\right]\left\{\begin{bmatrix} z_1 \\ \vdots \\ z_r \\ y \end{bmatrix} \,\middle|\, \dots\right.$$
$$\left. z_1,\dots,z_r \in \mathcal{C}^1_\infty,\ y \in \mathcal{C}^{q-r}_\infty,\ d_i\left(\frac{\partial}{\partial t}\right) z_i(t) = 0 \right\}$$

HA
$$\overset{\text{HA}}{=} T_1\left(\frac{\partial}{\partial t}\right)\left\{\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \in \mathcal{C}^r_\infty \,\middle|\, d_i\left(\frac{\partial}{\partial t}\right) z_i = 0\right\} \oplus \operatorname{image}_{\mathcal{C}_\infty}\left(T_2\left(\frac{\partial}{\partial t}\right)\right)$$

Analoguosly, we have

$$\mathfrak{L}_c(P) = \dots = T_1\left(\frac{\partial}{\partial t}\right)\left\{\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \in \mathcal{C}^r_c \,\middle|\, d_i\left(\frac{\partial}{\partial t}\right) z_i = 0\right\} \oplus \operatorname{image}_{\mathcal{C}_c}\left(T_2\left(\frac{\partial}{\partial t}\right)\right)$$

$= \operatorname{image}_{\mathcal{C}_c}\left(T_2\left(\frac{\partial}{\partial t}\right)\right)$, since for $d \in \mathbb{C}[\lambda]_k$ all solutions of $d\left(\frac{\partial}{\partial t}\right) z = 0$ are uniquely fixed by the „initial values"
$$z(t_0),\ z^{(1)}(t_0), \dots, z^{(k-1)}(t_0), \qquad (*)$$
where $t_0 \in \mathbb{R}$ can be choosen arbitrary. However, since $z \in \mathcal{C}^1_c$ there exists a $t_0 \in \mathbb{R}$ such that all values in $(*)$ are zero. Thus $z = 0$.

Remark: This shows that every compact behavior $\mathfrak{L}_c(P)$ admits an image representation. If $d_1,\dots,d_r$ are all nonzero constant then also $\mathfrak{L}(P)$ admits an image representation.

Zeros and Poles

Theorem 1.7. (Mac Millan form)
Let $R \in \mathbb{C}(\lambda)^{p,q}$ and set $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(R)$. Then there exist unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$, such that $R = S\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$, $D = \operatorname{diag}\left(\frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_r}{\beta_r}\right) \in \mathbb{C}(\lambda)^{r,r}$
(MF)

where $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \mathbb{C}[\lambda]$ with α_i, β_i for $i=1,\dots,r$ are unique, α_{i+1} divides α_i and β_i divides β_{i+1} for $i=1,\dots,r-1$.

Proof.:

Write $R = \frac{1}{d} N$ with $d \in \mathbb{C}[\lambda]$ being the least common multiple of all the entries in R and $N \in \mathbb{C}[\lambda]^{p,q}$. Then use the Smith form of N. $\blacksquare$

Definition 1.8.

Let the MacMillan form of $R \in \mathbb{C}(\lambda)^{p,q}$ be given by (MF), then we call

$\mathcal{Z}(R) := \{\lambda \in \mathbb{C} \mid \alpha_1(\lambda) \cdot \ldots \cdot \alpha_r(\lambda) = 0\}$ the zeros of R

and $\mathcal{P}(R) := \{\lambda \in \mathbb{C} \mid \beta_1(\lambda) \cdot \ldots \cdot \beta_r(\lambda) = 0\}$ the poles of R.

Also we call $\mathcal{D}(R) := \mathbb{C} \setminus \mathcal{P}(R)$ the domain of definition of R and assume w.l.o.g. that for every $\lambda_0 \in \mathcal{D}(R)$ the matrix $R(\lambda_0) \in \mathbb{C}^{p,q}$ is well defined.

For $P \in \mathbb{C}[\lambda]^{p,q}$ we have $\mathcal{P}(P) = \emptyset$, $\mathcal{D}(P) = \mathbb{C}$.

Lemma 1.9.: Let $R \in \mathbb{C}(\lambda)^{p,q}$. Then we have

$$\operatorname{rank}_{\mathbb{C}(\lambda)}(R) = \max_{\lambda_0 \in \mathcal{D}(R)} \underbrace{\operatorname{rank}}_{\text{in the usual sense}} R(\lambda_0)$$

Defining $\Lambda := \{\lambda_0 \in \mathcal{D}(R) : \operatorname{rank}(R(\lambda_0)) < \operatorname{rank}_{\mathbb{C}(\lambda)}(R)\}$ we have $\Lambda \subseteq \mathcal{Z}(R) \subseteq \Lambda \cup \mathcal{P}(R)$.

In particular, Λ is a finite set and for $P \in \mathbb{C}[\lambda]^{p,q}$ we have $\mathcal{Z}(P) = \Lambda$

Proof. Follows from the MacMillan form, since for unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ we have that $S(\lambda_0) \in \mathbb{C}^{p,p}$ is invertible (in the usual sense) for all $\lambda_0 \in \mathbb{C}$ (due to $\det S(\lambda_0) \equiv c \neq 0$) $\blacksquare$

Lemma 1.10.

Let $R \in \mathbb{C}(z)^{p,q}$ and $r := \operatorname{rank}_{\mathbb{C}(z)}(R)$. Then there exist polynomial kernel and co-kernel spanning matrices $U \in \mathbb{C}[z]^{q,q-r}$, $V \in \mathbb{C}[z]^{q,r}$ of R such that $[V, U]$ is even unimodular (not only invertible) and $\mathcal{Z}(R) = \mathcal{Z}(RV)$ and $\mathcal{P}(R) = \mathcal{P}(RV)$.

Proof: Consider the Mac Millan form and partition $T^{-1} =: [V, U]$ $\blacksquare$

Lemma 1.11.

Let $R \in \mathbb{C}(z)^{p,q}$ and $r := \operatorname{rank}_{\mathbb{C}(z)}(R)$. Let $U \in \mathbb{C}(z)^{q,q-r}$, $V \in \mathbb{C}(z)^{q,r}$ be kernel and co-kernel spanning matrices of R. In the Mac Millan form (MF) partition $T^{-1} =: [\underbrace{T_1}_{r \text{ cols}}, \underbrace{T_2}_{q-r}]$. Then there exist an invertible ~~U_2~~

$U_2 \in \mathbb{C}(z)^{q-r,q-r}$ with $\mathcal{P}(U) = \mathcal{P}(U_2)$, $\mathcal{Z}(U) = \mathcal{Z}(U_2)$, an invertible matrix $V_1 \in \mathbb{C}(z)^{r,r}$, and $V_2 \in \mathbb{C}(z)^{q-r,r}$ such that

$$[V, U] = T^{-1}\begin{bmatrix} V_1 & 0 \\ V_2 & U_2 \end{bmatrix} = [T_1 V_1 + T_2 V_2, T_2 U_2] \quad (*)$$

If U and V are polynomial then also U_2, V_1 and V_2 are polynomial.

Proof: The last sentence follows by premultiplying $(*)$ with T. For the first part set $\tilde{U} := TU$ and $\tilde{V} := TV$ and partition these matrices via

$$\tilde{U} := \begin{array}{c} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \end{array}\begin{array}{l} r \\ q-r \end{array} \quad \text{and} \quad \tilde{V} := \begin{array}{c} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \end{array}\begin{array}{l} r \\ q-r \end{array},$$

i.e., such that V_1 and U_2 are square.

Using the Mac Millan form, we find that

$$0 = S^{-1} R U = S^{-1} R T^{-1} T U = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$= \operatorname{diag}\left(\frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_r}{\beta_r}\right) \cdot U_1 \Rightarrow U_1 = 0$$

Since $[V, U]$ is invertible, U has full column rank; then $\tilde{U}$ has full column rank which means U_2 is invertible. From the relation $TU = \begin{bmatrix} 0 \\ U_2 \end{bmatrix}$ we see that U and U_2 have (in principle) the same Mac Millan form and thus the same zeros and poles. For $\tilde{V}$ we find

$$r = \operatorname{rank}_{\mathbb{C}(\lambda)} (P V) = \operatorname{rank}_{\mathbb{C}(\lambda)} (S^{-1} P T^{-1} T V)$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \right) = \operatorname{rank}_{\mathbb{C}(\lambda)} D V_1$$

D is invertible

$$\overset{\downarrow}{=} \operatorname{rank}_{\mathbb{C}(\lambda)} V_1$$ and thus that V_1 is invertible.

$\square$