Primeness

Definition 1.12.: Let $P \in \mathbb{C}[\lambda]^{p,q}$.

i) P is called *left prime* if $\operatorname{rank} P(\lambda_0) = p \ \forall \lambda_0 \in \mathbb{C}$.
ii) P is called *right prime* if $\operatorname{rank} P(\lambda_0) = q \ \forall \lambda_0 \in \mathbb{C}$.
iii) P is called *prime* if it is left prime or right prime.

Theorem 1.13.: [Zerz, Theorem 4.26]

$p \in \mathbb{C}[\lambda]^{1,1}$
$p(\lambda) = (\lambda-\lambda_1)\cdots(\lambda-\lambda_r)\cdot c$, $c \in \mathbb{C}$

For $P \in \mathbb{C}[\lambda]^{p,q}$ the following are equivalent:

a) P is left prime
b) P has the Smith form $S[I,0]T$.
c) $\mathcal{Z}(P) = \emptyset \wedge \operatorname{rank}_{\mathbb{C}(\lambda)}(P) = p$. Lemma 1.9. $\mathcal{Z}(P) \stackrel{!}{=} \{\lambda_0 \in \mathbb{C} \mid \operatorname{rank}_{\mathbb{C}(\lambda)}(P) > \operatorname{rank} P(\lambda_0)\}$
d) $p \leq q$ and there exists a Matrix $\tilde{P} \in \mathbb{C}[\lambda]^{(q-p),q}$ such that $\begin{bmatrix} P \\ \tilde{P} \end{bmatrix} \in \mathbb{C}[\lambda]^{q,q}$ is unimodular.
e) There exists a polynomial right inverse, i.e., a matrix $S \in \mathbb{C}^{q,p}$ with $PS = I$
(Any such S is then right prime).
f) If $P = UP_1$ for some $U \in \mathbb{C}[\lambda]^{p,p}$, $P_1 \in \mathbb{C}[\lambda]^{p,q}$, then U is unimodular.

Proof:

a) $\Leftrightarrow$ c): Lemma 1.9.

c) $\Leftrightarrow$ b): Follows from the Smith form

b) $\Rightarrow$ d): With the Smith form written as $P = S[I,0]\underbrace{T}_{=\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}}$
$= S[I,0]\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = ST_1$ set $\tilde{P} := T_2$ to obtain that

$$\begin{bmatrix} P \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} ST_1 \\ T_2 \end{bmatrix} = \left[\begin{array}{c|c} S & \\ \hline & I \end{array}\right]\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$ is a product of unimodular matrices.

d) ⇒ e): Unimodularity of $\begin{bmatrix} P \\ \tilde{P} \end{bmatrix}$ implies the existence of $X = [X_1, X_2] \in \mathbb{C}[\lambda]^{q,q}$ with

$$\begin{bmatrix} P \\ \tilde{P} \end{bmatrix} [X_1, X_2] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} PX_1 & PX_2 \\ \tilde{P}X_1 & \tilde{P}X_2 \end{bmatrix},$$

which implies $PX_1 = I$.

Now let S be an arbitrary polynomial matrix with $PS = I$. If there was a $\lambda_0 \in \mathbb{C}$ with $\operatorname{rank} S(\lambda_0) < p$. Then $p = \operatorname{rank} I_p = \operatorname{rank} P(\lambda_0) \cdot S(\lambda_0)$

$\leq \min\{\operatorname{rank} P(\lambda_0), \operatorname{rank} S(\lambda_0)\} \leq \operatorname{rank} S(\lambda_0) < p$ ↯

e) ⇒ f): Let S be with $PS = I$ and $P = UP_1$. Then $U \underbrace{P_1 S}_{= U^{-1} \in \mathbb{C}[\lambda]^{p,p}} = PS = I$ shows that U is unimodular.

f) ⇒ b): Assume to the contrary that the Smith form $P = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$ contains a nonconstant d_i on the diagonal of D or that $\operatorname{rank}_{\mathbb{C}(\lambda)} D < p$.

In both cases we have that in

$$P = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T = \underbrace{S \underbrace{\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}}_{\in \mathbb{C}[\lambda]^{p,p}}}_{=: U \in \mathbb{C}[\lambda]^{p,p}} \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{\in \mathbb{C}^{p,q}} T$$

the matrix U is not unimodular ▨

An analogues theorem holds for right prime matrices.

Corollary 1.14.

For every $R \in \mathbb{C}(\lambda)^{p,q}$ there exists a polynomial (right) prime kernel spanning matrix and a polynomial (right) prime cokernel spanning matrix.

Proof: Use theorem 1.13. d) (rewritten for right prime matrices) and Lemma 1.10. ▨ (Homework)

Elimination of latent variables

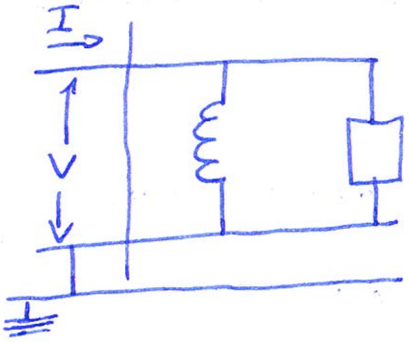
Definition 1.15.: Let $P \in \mathbb{C}[\lambda]^{p,q}$, $M \in \mathbb{C}[\lambda]^{p,r}$. Then the behavior $\mathfrak{B} := \{ z \in \mathcal{C}_\infty^q \mid \exists\, \ell \in \mathcal{C}_\infty^r \text{ with } P(\tfrac{\partial}{\partial t}) z = M(\tfrac{\partial}{\partial t}) \ell \}$ is called a latent variable description.

In this case, ℓ is called the latent variable and z is called the manifest variable.

Furthermore,

$$\mathfrak{B}_f := \left\{ \begin{bmatrix} z \\ \ell \end{bmatrix} \in \mathcal{C}_\infty^{q+r} \;\middle|\; P(\tfrac{\partial}{\partial t}) z = M(\tfrac{\partial}{\partial t}) \ell \right\} = \mathfrak{B}([P, -M])$$

is called the associated full behavior. In contrast to this $\mathfrak{B}$ is called the manifest behavior



$$P(\lambda) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -R & & 1 & 0 \\ & \lambda L & & 1 \end{bmatrix} \qquad z = \left[\begin{array}{c} I_R \\ I_L \\ I \\ V \end{array}\right] \begin{array}{l} \text{latent} \\ \\ \text{manifest} \\ \\ \end{array}$$

Theorem 1.16.: For every system $\mathfrak{B}$ in latent variable description there exists a $\tilde{P} \in \mathbb{C}[\lambda]^{\tilde{p},q}$ such that the manifest behavior is given by $\mathfrak{B} = \mathfrak{B}(\tilde{P})$.

Proof: We use the notation of Definition 1.15.

Let $U \in \mathbb{C}[\lambda]^{p,p}$ be unimodular such that

$UM = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$, where M_1 has full row rank (Existence of U: Homework)

Partition $UP =: \begin{bmatrix} P_1 \\ \tilde{P} \end{bmatrix}$ accordingly and observe that then

$$\mathcal{L}_e = \{ z \mid \exists e \quad U(\tfrac{\partial}{\partial t}) P(\tfrac{\partial}{\partial t}) z = U(\tfrac{\partial}{\partial t}) M(\tfrac{\partial}{\partial t}) e \}$$

$$= \{ z \mid \exists e \quad \begin{bmatrix} P_1(\frac{\partial}{\partial t}) z \\ \tilde{P}(\frac{\partial}{\partial t}) z \end{bmatrix} = \begin{bmatrix} M_1(\frac{\partial}{\partial t}) e \\ 0 \end{bmatrix} \} \subseteq \mathcal{L}_e(\tilde{P})$$

The other inclusion $\mathcal{L}_e \supseteq \mathcal{L}_e(\tilde{P})$ follows from the following Lemma 1.17. a)

$\blacksquare$

Lemma 1.17.: Let $M \in \mathbb{C}[\lambda]^{p,q}$. Then we have the following: a) M has full row rank if and only if: for every $y \in \mathcal{C}^p_\infty$ there exists an $e \in \mathcal{C}^q_\infty$ such that $M(\frac{\partial}{\partial t}) e(t) = y(t) \ \forall t \in \mathbb{R}$

b) M has full collumn rank if and only if: there exists an open interval $\mathbb{I} \subseteq \mathbb{R}$ such that for every $e \in \mathcal{C}^q_\infty$ which vanishes identically on $\mathbb{I}$ and satisfies $M(\frac{\partial}{\partial t}) e(t) = 0 \ \forall t \in \mathbb{R}$ we have $e = 0$.

Proof: a) „$\Rightarrow$" In this case the Smith form is $M = S[D, 0]T$. Let $y \in \mathcal{C}^p_\infty$ be arbitrary. Since

$$(*) \quad [Me = y] \Leftrightarrow [S[D,0]\underbrace{Te}_{=:\tilde{e}} = y] \Leftrightarrow [[D,0]\underbrace{\tilde{e}}_{=\begin{bmatrix} e_1 \\ \vdots \\ e_q \end{bmatrix}} = S^{-1} y] \Big\}$$

we set $\tilde{y} := S^{-1}(\frac{\partial}{\partial t}) y$ and denote the entries of $\tilde{y}$ by $\tilde{y} =: \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$. (Remember: $D =: \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{bmatrix}$)

From the theory of ODEs (or simply by dividing by a nonzero constant if d_i is a nonzero constant) we know that the scalar equation

$$d_i\left(\tfrac{\partial}{\partial t}\right) e_i = y_i \qquad i=1,\dots,p$$

all have (at least) one solution.

In the following let $e_1,\dots,e_p$ denote (any one of) those solutions. Set $\tilde{e} := \begin{bmatrix} e_1 \\ \vdots \\ e_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathcal{C}^q_\infty$ and with this $e = T^{-1}\left(\tfrac{\partial}{\partial t}\right)\tilde{e}$.

Then by (*) we obtain that $M\left(\tfrac{\partial}{\partial t}\right) e = y$.

„<=" We show the negation, i.e., assume M has not full row rank which means that the Smith form is $M = S\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$ $\}\geq 1$ rows.

Set $\hat{y}(t) := S\left(\tfrac{\partial}{\partial t}\right)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ ← last element

To show that there is no $\hat{e}$ with $M\left(\tfrac{\partial}{\partial t}\right)\hat{e} = \hat{y}$ we can equivalently show that there is no $\tilde{e}$ with

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\tilde{e} = S^{-1}\left(\tfrac{\partial}{\partial t}\right)\hat{e} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$ (this equivalence follows similar to (*)).

Such an $\tilde{e}$ can clearly not exist, since the last entry in $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\tilde{e}$ is always zero.

b) „=>" In this case the Smith form is $M = S\begin{bmatrix} D \\ 0 \end{bmatrix} T$.

Thus every $e \in \mathcal{C}^q_\infty$ with $M\left(\tfrac{\partial}{\partial t}\right) e = 0$ satisfies

$$0 = (S^{-1}M)\left(\tfrac{\partial}{\partial t}\right) e = \begin{bmatrix} D\left(\tfrac{\partial}{\partial t}\right) \\ 0 \end{bmatrix}\underbrace{T\left(\tfrac{\partial}{\partial t}\right) e}_{=:\tilde{e} = [e_1,\dots,e_q]^T}$$ and all the $e_1,\dots,e_q$ also vanish identically in $\mathbb{I}$.

Thus there exists a (or: Thus for every) $t_0 \in \mathbb{I}$ such that

$e_i^{(j)}(t_0) = 0 \quad \forall\, i = 1, \dots, q \ , j \in \mathbb{N}_0$.

Since, however, $d_i\left(\frac{\partial}{\partial t}\right) e_i = 0 \quad i = 1, \dots, q$ we conclude from the standart theory of ODEs that $e_1 = \dots = e_q = 0$

$\Rightarrow \tilde{e} = 0 \Rightarrow e = T^{-1}\left(\frac{\partial}{\partial t}\right) \tilde{e} = 0$

„<=" Assume to the contrary that M does not have full column rank. Then by Corollary 1.14 there exists a prime kernel spanning matrix $U \in \mathbb{C}[\lambda]^{q,m}$ which in this case satisfies $m \geq 1$.

According to Theorem 1.13. e) (formulated for right prime matrices), let $\tilde{U}$ be with $\tilde{U} U = I$.

Let $\alpha \in \mathcal{C}_\infty^m$ be a nontrivial function which vanishes in $\mathbb{I}$. Then $z := U\left(\frac{\partial}{\partial t}\right) \alpha \in \mathcal{C}_\infty^q$ vanishes in $\mathbb{I}$ and fullfills $M\left(\frac{\partial}{\partial t}\right) z = \underbrace{(MU)}_{\substack{= 0, \text{ since } U \text{ is kernel spanning} \\ \text{matrix of } M.}}\left(\frac{\partial}{\partial t}\right) \alpha = 0$

although $\tilde{U}\left(\frac{\partial}{\partial t}\right) z = (\tilde{U} U)\left(\frac{\partial}{\partial t}\right) \alpha = \alpha \neq 0$, which implies $z \neq 0$. ▨

Autonomous behaviors

Definition 1.18. Let $R \in \mathbb{C}[\lambda]^{p,q}$.

a) $\mathcal{L}e(R)$ is called autonomous if all trajectories are determined by their past, i.e.,

$$[z_1, z_2 \in \mathcal{L}e(R) \text{ with } z_1(t) = z_2(t),\ t \le 0] \Rightarrow [z_1 = z_2]$$

b) If there exists a permutation $\Pi \in \mathbb{R}^{q,q}$ such that

$$R\Pi = [\underbrace{P}_{e\text{ cols}} \,\vdots\, \underbrace{Q}_{m\text{ cols}}] \text{ then}$$

(P, Q) is called a <u>partition of R</u>. In this case we split up the original space $\Pi^{-1} z = \begin{bmatrix} y \\ \hdashline u \end{bmatrix} \begin{matrix} e\text{-elements} \\ m \text{ elements} \end{matrix}$

accordingly, such that $[R(\frac{\partial}{\partial t}) z(t) = 0] \Leftrightarrow [P(\frac{\partial}{\partial t}) y(t) \ldots + Q(\frac{\partial}{\partial t}) u(t) = 0]$.

c) For a partition (P, Q) with signal space $\begin{bmatrix} y \\ u \end{bmatrix}$ we say that <u>u is free</u> if for all $u \in \mathcal{C}^m_\infty$ there exists a $y \in \mathcal{C}^e_\infty$ such that $\begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{L}e([P, Q])$.

In this case we say that $\mathcal{L}e(R)$ has free components.

Theorem 1.19: Let $R \in \mathbb{C}[\lambda]^{p,q}$. Then the following are equivalent: a) R has full column rank

b) $\mathcal{L}e(R)$ is autonomous

c) $\mathcal{L}e(R)$ has no free components

Proof: One first shows that $\mathcal{L}e(R)$ is autonomous if and only if $[z \in \mathcal{L}e(R),\ z(t) = 0\ \forall t \le 0] \Rightarrow [z = 0]$ (HA)

a) ⇒ b) Let $z \in \mathcal{L}e(R)$; $z(t) = 0\ \forall t \le 0$. Using Lemma 1.17 b) we conclude $z = 0$

b) => a)

Again follows from Lemma 1.17.b). $\mathbb{I} := (-\infty, 0]$

c) => a) Assume to the contrary that R does not have full column rank. Denote the columns of R by $r_1, \dots, r_q \in \mathbb{C}[z]^{p,1}$. Since

$$\operatorname{range}_{\mathbb{C}(z)} R = \operatorname{span}_{\mathbb{C}(z)}(r_1, \dots, r_q)$$

one can select a basis of the linear space (Basisauswahlsatz) $\operatorname{range}_{\mathbb{C}(z)} R$ out of $r_1, \dots, r_q$. Let Π be the permutation which moves those basis elements to the front. Then we have

$$R\Pi = [\underbrace{P}_{\substack{q-m \\ \text{cols}}} \mid \underbrace{Q}_{\substack{m \geq 1 \\ \text{cols}}}]$$ where P contains the basis

elements, i.e., P has full column rank and

$$\operatorname{range}_{\mathbb{C}(z)} R = \operatorname{range}_{\mathbb{C}(z)} P \Rightarrow \operatorname{rank}_{\mathbb{C}(z)} R = \operatorname{rank}_{\mathbb{C}(z)} P \overset{\substack{P \text{ has full} \\ \text{column rank}}}{=} q-m.$$

Let $U \in \mathbb{C}[z]^{p,p}$ be unimodular with $UR = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ where R_1 has full row rank $q-m$.

$$\Rightarrow \underbrace{U[P, Q]}_{= R\Pi} =: \begin{array}{cl} \left[\begin{array}{c|c} P_1 & Q_1 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} q-m \text{ rows} \\ \\ \end{array} \end{array}, \quad R_1 \Pi = [P_1, Q_1].$$

Since $UP = \begin{bmatrix} P_1 \\ 0 \end{bmatrix}$, P_1 still has full column rank $q-m$. Thus P_1 is invertible. Using Lemma 1.17. a) this means that for every $u \in \mathcal{C}_\infty^{q-m}$ there exists a $y \in \mathcal{C}_\infty^{q-m}$ with $P_1(\frac{\partial}{\partial t})y = -Q(\frac{\partial}{\partial t})u$

$$\Rightarrow [P(\tfrac{\partial}{\partial t}), Q(\tfrac{\partial}{\partial t})]\begin{bmatrix} y \\ u \end{bmatrix} = 0$$ which shows that u is free.

a) ⇒ c) By contradiction: Let $(\tilde{P}, \tilde{Q})$ be any partition of R with signal space $\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix}$, where $\tilde{Q}$ has at least one column. $\tilde{e}, \tilde{m}$ cols

We show that $\tilde{u}$ can not be free. Let U be unimodular with $U[\tilde{P}, \tilde{Q}] = \begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix}$ such that P_1 and Q_2 are invertible (Existence: Homework)

Thus for all trajectories $(\tilde{y}, \tilde{u}) \in \mathcal{L}e([\tilde{P}, \tilde{Q}])$
$= \mathcal{L}e\left(\begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix}\right)$ we have $Q_2(\frac{\partial}{\partial t})\tilde{u} = 0$.

However, using Lemma 1.17 a) we can construct a $\tilde{u}_0$ with $Q_2(\frac{\partial}{\partial t})\tilde{u}_0 \neq 0$. For this $\tilde{u}_0$ we then have that

$$(\tilde{y}, \tilde{u}_0) \notin \mathcal{L}e\left(\begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{L}e([\tilde{P}, \tilde{Q}]) \text{ for all } \tilde{y}.$$

Thus $\tilde{u}$ is not free. $\blacksquare$