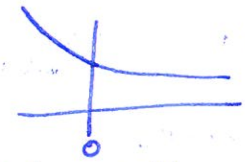
Remind: $P \in \mathbb{C}[\lambda]^{p,q}$, $\mathcal{L}(P)$ autonomous



$$P\left(\frac{\partial}{\partial t}\right) z = 0, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_q \end{bmatrix} \xrightarrow{\pi} \begin{array}{c} \left[\begin{array}{c} z_2 \\ z_3 \\ \hline z_1 \\ z_4 \end{array}\right] \end{array} \begin{array}{l} y \\ \\ u \end{array}$$

u is free $:\Leftrightarrow \forall u \in \mathcal{C}^{\omega}_{\infty} \; \exists y$ s.t. $(y,u) \in \mathcal{L}(P\pi)$

Input / Output behaviors

Definition 1.20.

Let $R \in \mathbb{C}[\lambda]^{p,q}$. In a partition (P,Q) of R with signal space $\begin{bmatrix} y \\ \hline u \end{bmatrix} \begin{matrix} e \\ m \end{matrix}$ we call u input and y output if u is maximally free. This means that u is free and there exists no other partition $(\tilde{P}, \tilde{Q})$ of R with signal space $\begin{bmatrix} \tilde{y} \\ \hline \tilde{u} \end{bmatrix} \begin{matrix} \tilde{e} \\ \tilde{m} \end{matrix}$ in which $\tilde{u}$ is free such that $\tilde{m} > m$, i.e., no other partition with more free components.

In this case we call (P,Q) input/output.

Theorem 1.21:

Let (P,Q) be a partition of $R \in \mathbb{C}[\lambda]^{p,q}$. Then (P,Q) is input/output if and only if $\operatorname{rank}_{\mathbb{C}(\lambda)} R = \operatorname{rank}_{\mathbb{C}(\lambda)} P$ and P has full column rank.

Proof: Let $\begin{bmatrix} y \\ \hline u \end{bmatrix} \begin{matrix} e \\ m \end{matrix}$ be the signal space of (P,Q).

"$\Rightarrow$" If P would not have full column rank then by Theorem 1.19. there would be free components in y which we could add to the elements of u. Then u would not be maximally free. ↯ Thus P has full column rank.

If $\operatorname{rank}_{\mathbb{C}(\lambda)} P < \operatorname{rank}_{\mathbb{C}(\lambda)}[P, Q] = \operatorname{rank}_{\mathbb{C}(\lambda)} R$ then there

($\square$) exists a unimodular U with $U[P\,Q] = \begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix}$ where

P_1 and Q_2 have full row rank. (Homework).

Use Lemma 1.17. a) to construct a $u_0 \in \mathcal{C}_\infty^m$ with

$Q_2(\frac{\partial}{\partial t}) u_0(t) \neq 0$.

Since $\mathcal{L}e([P,Q]) = \mathcal{L}e\left(\begin{bmatrix} P_1 & Q_1 \\ 0 & Q_2 \\ 0 & 0 \end{bmatrix}\right)$ we have for all $y \in \mathcal{C}_\infty^e$ that

($\square$) $\begin{bmatrix} y \\ u_0 \end{bmatrix} \notin \mathcal{L}e([P,Q])$. This shows that u is not free. ↯

Thus $\operatorname{rank}_{\mathbb{C}(\lambda)} R = \operatorname{rank}_{\mathbb{C}(\lambda)} P$.

„$\Leftarrow$" Since $\operatorname{rank}_{\mathbb{C}(\lambda)} R = \operatorname{rank}_{\mathbb{C}(\lambda)} P$ there exists a unimodular

Matrix U such that $U[P,Q] = \begin{bmatrix} P_1 & Q_1 \\ 0 & 0 \end{bmatrix}$ with P_1 having

full row rank.

Let $u \in \mathcal{C}_\infty^m$ be arbitrary. By Lemma 1.17. a) there

exists a $\underset{\mathcal{C}_\infty^e}{\overset{\in}{y}}$ with $P_1(\frac{\partial}{\partial t}) y(t) = -Q(\frac{\partial}{\partial t}) u(t)$

$\Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{L}e\left(\begin{bmatrix} P_1 & Q_1 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{L}e([P,Q])$.

Since u was arbitrary, u is free.

Let $(\tilde{P}, \tilde{Q})$ be an arbitrary partition of R with signal space

$\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} \begin{matrix} \tilde{e} \\ \tilde{m} \end{matrix}$ such that $\tilde{u}$ is free. As above at ($\square$) one shows

by assuming the contrary that then $\operatorname{rank}_{\mathbb{C}(\lambda)} \tilde{P} = \operatorname{rank}_{\mathbb{C}(\lambda)} R$.

Using the assumption $\operatorname{rank}_{\mathbb{C}(\lambda)} P = e$ this implies

$\operatorname{rank}_{\mathbb{C}(\lambda)} \tilde{P} = \operatorname{rank}_{\mathbb{C}(\lambda)} R = \operatorname{rank}_{\mathbb{C}(\lambda)} P = e \Rightarrow \tilde{e} \underset{1}{\geq} e$

The rank is less or equal to the number of columns in $\tilde{P}$.

$\Rightarrow \tilde{m} = q - \tilde{\ell} \leq q - \ell = m$ ☒

Corollary 1.22.

Every $R \in \mathbb{C}[z]^{p,q}$ admits an input/output partition.

Proof: Denote the columns of R by $r_1, \dots, r_q \in \mathbb{C}[z]^p$.
Since $\mathrm{range}_{\mathbb{C}(z)} R = \mathrm{span}_{\mathbb{C}(z)}(r_1, \dots, r_q)$ we can select a basis of $\mathrm{range}_{\mathbb{C}(z)} R$ out of $r_1, \dots, r_q$. Let $\Pi \in \mathbb{R}^{q,q}$ be the permutation which moves the basis to the front and set $R\Pi =: [P, Q]$.

Then $\mathrm{range}_{\mathbb{C}(z)} R = \mathrm{range}_{\mathbb{C}(z)} P$ and P has full column rank (over $\mathbb{C}(z)$) $\Rightarrow \mathrm{rank}_{\mathbb{C}(z)} R = \mathrm{rank}_{\mathbb{C}(z)} P$ and P has full column rank (over $\mathbb{C}(z)$) sinc its colums form a basis. ☒

Linear time varying state-space systems

Let $\mathcal{C}_\infty^{n,m}$ denote the infinitely often differentiable matrix-valued functions $G: \mathbb{R} \to \mathbb{C}^{n,m}$

In basic ODE courses one shows that for $A \in \mathcal{C}_\infty^{n,n}$ and $B \in \mathcal{C}^{n,m}$ the linear time-varying state-space system

$$(\text{LTVS}) \quad \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x_0 = x(0) \end{cases}$$

has a unique solution $x \in \mathcal{C}_\infty^n$ for every $x_0 \in \mathbb{C}^n$ (Picard - Lindelöff).

Especially, we can denote by $x_i(t,t_0)$ the unique solution of

$$\begin{cases} \dot{x}(t) = A(t)x(t) \\ x(t_0) = e_i \quad \leftarrow \text{i-the unit-vector} \end{cases}$$

evaluated at $t \in \mathbb{R}$ and then define $\Phi: \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{n,n}$ by $\Phi(t,t_0) := [x_1(t,t_0), \dots, x_n(t,t_0)]$.

Then
$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,t_0) &= \left[\frac{\partial}{\partial t}x_1(t,t_0), \dots, \frac{\partial}{\partial t}x_n(t,t_0)\right] \\ &= [A(t)x_1(t,t_0), \dots, A(t)x_n(t,t_0)] \\ &= A(t)[x_1(t,t_0), \dots, x_n(t,t_0)] = A(t)\Phi(t,t_0) \end{aligned}$$

and $\Phi(t_0,t_0) = [e_1, \dots, e_n] = I$.

<u>Definition 1.23:</u> Let $A \in \mathcal{C}_\infty^{n,n}$. Then the infinitely often differentiable function $\Phi: \mathbb{R}\times\mathbb{R} \to \mathbb{C}^{n,n}$ which is given by the unique solution of the matrix ODE(s) $\leftarrow$ (it is a different ODE for every $t_0 \in \mathbb{R}$)

$$\begin{cases} \frac{\partial}{\partial t} \Phi(t,t_0) = A(t)\Phi(t,t_0) \\ \Phi(t_0,t_0) = I \end{cases}$$

is called the fundamental matrix of $\dot{x} = A(t)x$.

Theorem 1.24: For $x_0 \in \mathbb{C}^n$ and $u \in \mathcal{C}^m_\infty$ the unique solution $x \in \mathcal{C}^n_\infty$ of (LTVS) is given by the variation of constants formula

$$\text{(VC)} \quad x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)\,ds$$

where Φ is the fundamental matrix of $\dot{x} = A(t)x$.

Proof: Compute $\dot{x}$ by using $\frac{\partial}{\partial t}\left(\int_{t_0}^{t} f(t,s)\,ds\right)$

$$= f(t,t) + \int_{t_0}^{t} \frac{\partial}{\partial t} f(t,s)\,ds$$

$\blacksquare$

Choosing the universe $\mathcal{U} := \mathcal{C}^{n+m}_\infty$ the (not time-invariant) behavior of (LTVS) can (for any fixed $t_0 \in \mathbb{R}$) be defined by

$$\mathcal{L} := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{U} \;\middle|\; \exists x_0 \in \mathbb{C}^n \text{ such that (VC) holds} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{U} \;\middle|\; \dot{x}(t) = A(t)x(t) + B(t)u(t) \right\}$$

If $A(t) \equiv A \in \mathbb{C}^{n,n}$ and $B(t) \equiv B \in \mathbb{C}^{n,m}$ are constant in time, then $\mathcal{L}$ is time-invariant. In this case the fundamental matrix of $\dot{x} = Ax$ can be written in the form $\Phi(t,t_0)$

$$= e^{(t-t_0)A} := \sum_{i=0}^{\infty} \frac{(t-t_0)^i}{i!} A^i$$

and thus we have $\mathcal{L}e = \mathcal{L}e([\lambda I - A, -B])$

$$= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{U} \mid \exists x_0 \in \mathbb{C}^n \text{ such that } x(t) = e^{t \cdot A} x_0 + \int_0^t e^{(t-s)A} Bu(s)\,ds \right\}$$

Lemma 1.24:

Let $A \in C_\infty^{n,n}$ and let $\Phi(t,t_0)$ be the fundamental matrix of $\dot{x} = A(t)x$. Then the following hold:

1) $\Phi(t,s) \cdot \Phi(s,u) = \Phi(t,u) \quad \forall t,s,u \in \mathbb{R}$

2) $(\Phi(t,s))^{-1} = \Phi(s,t) \quad \forall t,s \in \mathbb{R}$

3) $\Psi(t,t_0) := \Phi^*(t_0,t)$ is the fundamental matrix of the so called adjoint system $\dot{z}(t) = -A^*(t)z$.

Proof: 1) can be shown with the uniqueness of solutions, see ODE course.

2) $I = \Phi(t,t) \overset{1.}{=} \Phi(t,s)\Phi(s,t)$

3) Since $0 = \frac{\partial}{\partial s}(I) \overset{2)}{=} \frac{\partial}{\partial s}\left(\Phi(s,t)\Phi(t,s)\right)$

product rule $\overset{\downarrow}{=} (D_1\Phi(s,t))\Phi(t,s) + \Phi(s,t)(D_2\Phi(t,s))$

$$= A(s)\underbrace{\Phi(s,t)\Phi(t,s)}_{=I} + \Phi(s,t)(D_2\Phi(t,s)),$$

where D_i denotes the derivative with respect to the i-th variable, we have

$\Phi(s,t)(D_2\Phi(t,s)) = -A(s) \Rightarrow D_2\Phi(t,s) = -(\Phi(s,t))^{-1}A(s)$

$\overset{2)}{=} -\Phi(t,s)A(s) \Rightarrow D_2\Phi^*(t,s) = -A^*(s)\underbrace{\Phi^*(t,s)}_{=:\Psi(s,t)}$

$\Rightarrow \frac{\partial}{\partial s}\Psi(s,t) = -A^*(s)\Psi(s,t)$

$\blacksquare$

Chapter 2: Controllability, Stabilizability, Observability, Reconstructability

Controllability of LTVS-systems

Definition 2.1:

Let $A \in \mathcal{C}_\infty^{n,n}$ and $B \in \mathcal{C}_\infty^{n,m}$ and $t_0 < t_1$. Then the system

(LTVS) $\quad \dot{x}(t) = A(t)x(t) + B(t)u(t)$

is called controllable from t_0 to t_1 if for all $x_0, x_1 \in \mathbb{C}^n$ there exists a $u \in \mathcal{C}_\infty^m$ such that the unique solution $x \in \mathcal{C}_\infty^n$ of (LTVS) with $x(t_0) = x_0$ satisfies $x(t_1) = x_1$.

Theorem 2.2.:

Let $A \in \mathcal{C}_\infty^{n,n}$, $B \in \mathcal{C}_\infty^{n,m}$ and $t_0 < t_1$. Let Φ be the fundamental matrix of $\dot{x} = A(t)x$ and define by

$$V(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, s) B(s) B^*(s) \Phi^*(t_0, s)\, ds$$

the Gramian of controllability.

Then the following are equivalent:

1) (LTVS) is controllable from t_0 to t_1.
2) Every solution $z \in \mathcal{C}_\infty^n$ of $\begin{cases} \dot{z}(t) = -A^*(t)z \\ 0 = B^*(t)z \end{cases}$ vanishes identically in $[t_0, t_1]$.
3) The Gramian of controllability is positive definite: $V(t_1, t_0) > 0$

Proof: 1) $\Rightarrow$ 2) Assume to the contrary that there was a $\hat{z} \in \mathcal{C}_\infty^n$ with $\dot{\hat{z}} = -A^*(t)\hat{z}$, $0 = B^*(t)\hat{z}$ but $\hat{z}(\hat{t}) \neq 0$ for some $\hat{t} \in [t_0, t_1]$. Using Lemma 1.25.c) we conclude that $\hat{z}(t_0) = \underbrace{\Phi^*(\hat{t}, t_0)}_{\text{invertible by Lemma 1.25 b)}} \hat{z}(\hat{t}) \neq 0$

Set $x_0 := \hat{z}(t_0)$. Using the assumption 1) we obtain the existence of a $\hat{u} \in \mathcal{C}_\infty^m$ such that the associated solution $\hat{x} \in \mathcal{C}_\infty^n$ of (LTVS) with $\hat{x}(t_0) = x_0$ satisfies $\hat{x}(t_1) = 0 \; (=: x_1)$.

Thus the function $f: \mathbb{R} \to \mathbb{R}$, $f(t) := \hat{x}^*(t)\hat{z}(t)$ satisfies

$$f(t_0) = \hat{x}^*(t_0)\,\hat{z}(t_0) = \|\hat{z}(t_0)\|_2^2 = \|x_0\|_2^2 \neq 0 \text{ and}$$

$$f(t_1) = \hat{x}^*(t_1)\hat{z}(t_1) = 0^* \cdot (\hat{z}(t_1)) = 0.$$

This, however, is a contradiction to

$$\begin{aligned}\frac{\partial}{\partial t} f(t) &= \dot{\hat{x}}^*(t)\hat{z}(t) + \hat{x}^*(t)\,\dot{\hat{z}}(t) \\ &= (A(t)\hat{x} + B(t)\,\hat{u}(t))^*\hat{z}(t) + \hat{x}^*(t)(-A^*(t)\hat{z}(t)) \\ &= \hat{x}^*(t)A^*(t)\hat{z}(t) + \hat{u}^*(t)\underbrace{B^*(t)\,\hat{z}(t)}_{=0} \\ &\cancel{=} \hat{x}^*(t)\,A^*(t)\hat{z}(t) = 0.\end{aligned}$$