Thm 2.13: Equivalent are: $(P \in \mathbb{C}[\lambda]^{p,q})$

1) $\mathfrak{L}(P)$ is controllable from t_0 to t_1

2) $\mathcal{Z}(P) = \emptyset$

3) $\exists\, U \in \mathbb{C}[\lambda]^{q,m}$ s.t. $\mathfrak{L}(P) = \text{image}_{\mathcal{C}_\infty}(U(\frac{\partial}{\partial t}))$

4) $\forall\, U \in \mathbb{C}[\lambda]^{p,m}$ right prime kernel spanning matrices we have $\mathfrak{L}(P) = \text{image}_{\mathcal{C}_\infty}(U(\frac{\partial}{\partial t}))$

Proof: 1) => 2) Let $P = S\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$, $D \in \mathbb{C}[\lambda]^{r,r}$ be the Smith form. Since $\mathfrak{L}(P)$ is controllable from t_0 to t_1 if and only if $\mathfrak{L}(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix})$ is controllable from 0 to $t_1 - t_0 =: \tau\ (>0)$ (Homework) we can as well assume the latter.

To show 2) assume to the contrary that there exists a $\lambda_0 \in \mathcal{Z}(P) \underset{\text{Def}}{=} \mathcal{Z}(D)$, i.e, that there exists an $i \in \{1, \dots, r\}$ such that
$$d_i(\lambda_0) = 0 \qquad \text{where } D =: \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{bmatrix}.$$

Set $y_i(t) := e^{\lambda_0 t}$ and conclude (Homework) that
$$d_i(\tfrac{\partial}{\partial t})\, y_i(t) \overset{!}{=} \underbrace{d_i(\lambda_0)}_{=0}\, y_i(t) = 0$$

Set $z_0 := \begin{bmatrix} \vdots \\ 0 \\ y_i \\ 0 \\ \vdots \end{bmatrix} \in \mathcal{C}_\infty^q$ $\leftarrow$ i-th element and $z_1 := 0 \in \mathcal{C}_\infty^q$, so that

$z_0, z_1 \in \mathfrak{L}(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix})$. By assumption there exists a $z \in \mathfrak{L}(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix})$ such that $z(t) = \begin{cases} z_0(t), & t \le 0 \\ z_1(t), & t \ge \tau \end{cases}$.

Denoting the i-th component of z by $\tilde{y}_i := z|_i$ we find that $\tilde{y}_i(t) = \begin{cases} e^{\lambda t}, & t \leq 0 \\ 0, & t \geq \tau \end{cases} = \begin{cases} z_0|_i, & t \leq 0 \\ z_1|_i, & t \geq \tau \end{cases}$ solves

$$d_i(\tfrac{\partial}{\partial t})\tilde{y}_i(t) = 0.$$

Thus we have $y_i, \tilde{y}_i \in \mathfrak{L}(d_i)$ with $y_i(t) = \tilde{y}_i(t)$ for all $t \leq 0$ although $y_i \neq \tilde{y}_i$. This is a contradiction, since by Theorem 1.19. $\mathfrak{L}(d_i)$ is autonomous ($d_i \neq 0$).

<u>2) ⇒ 3)</u> In this case the Smith form is $P = S\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}T$. Partition the inverse of T accordingly as $T^{-1} = [\underbrace{T_1}_{r\,\text{cols}} \mid \underbrace{T_2}_{q-r\,\text{cols}}]$. Then $\mathfrak{L}(P) = \mathfrak{L}(S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}T)$

$$= T^{-1}(\tfrac{\partial}{\partial t})\,\mathfrak{L}\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= [T_1(\tfrac{\partial}{\partial t}), T_2(\tfrac{\partial}{\partial t})]\left\{\begin{bmatrix} 0 \\ z_2 \end{bmatrix} \;\middle|\; z_2 \in \mathcal{C}_\infty^{q-r}\right\} = \operatorname{image}_{\mathcal{C}_\infty}(T_2(\tfrac{\partial}{\partial t}))$$

<u>3) ⇒ 4)</u> if U is a right prime polynomial kernel spanning matrix. Then with the notation from the previous inclusion [2) ⇒ 3)] we obtain with Lemma 1.11 that there exists an invertible $U_2 \in \mathbb{C}[\lambda]^{q-r,q-r}$, such that $U = T_2U_2$ and

$$\mathcal{Z}(U_2) = \mathcal{Z}(U) \overset{\text{Thm 1.13. c)}}{=} \emptyset.$$

Thus U_2 is unimodular and we have

$$\mathfrak{L}(P) \overset{\text{see above}}{=} \operatorname{image}_{\mathcal{C}_\infty}(T_2(\tfrac{\partial}{\partial t})) = \operatorname{image}_{\mathcal{C}_\infty}(T_2U_2)(\tfrac{\partial}{\partial t})$$

$$= \operatorname{image}_{\mathcal{C}_\infty}(U(\tfrac{\partial}{\partial t}))$$

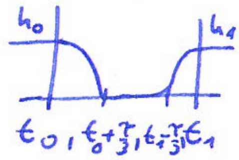
4) $\Rightarrow$ 1) Let $U \in \mathbb{C}[\lambda]^{q-r,q-r}$ be any right prime kernel spanning matrix (Existence: Lemma 1.10).
Then $\mathfrak{L}(P) = \mathrm{image}_{\mathcal{C}_\infty}(U(\frac{\partial}{\partial t}))$.

Let $z_0, z_1 \in \mathfrak{L}(P)$. Then there exists $\alpha_0, \alpha_1 \in \mathcal{C}_\infty^{q-r}$ such that

$$z_0 = U(\tfrac{\partial}{\partial t})\alpha_0, \quad z_1 = U(\tfrac{\partial}{\partial t})\alpha_1 .$$

Set $t_1 - t_0 =: \tau$. Let $h_0, h_1 \in \mathcal{C}_\infty^1$ be with

$$h_0(t) = \begin{cases} 1 & , t \leq t_0 \\ 0 & , t \geq t_0 + \frac{\tau}{3} \end{cases}, \quad h_1(t) = \begin{cases} 0, & t \leq t_1 - \frac{\tau}{3} \\ 1, & t \geq t_1 \end{cases}.$$



Define $\beta(t) := h_0(t)\alpha_0(t) + h_1(t)\alpha_1(t)$ such that $\beta \in \mathcal{C}_\infty^{q-r}$ and $\beta(t) = \begin{cases} \alpha_0(t) & , t \leq t_0 \\ \alpha_1(t) & , t \geq t_1 \end{cases}$.

Then $z(t) := U(\frac{\partial}{\partial t})\beta(t) = \begin{cases} z_0(t) & , t \leq t_0 \\ z_1(t) & , t \geq t_1 \end{cases}$ and controllability from t_0 to t_1 is shown.

The equivalence of 5) and 2) follows from Lemma 4 on the handout about the Kronecker canonical form.

$\blacksquare$

The previous Theorem shows that, for systems $\mathfrak{L}(P)$ with $P \in \mathbb{C}[\lambda]^{p,q}$ the notation of controllability does not depend on t_0 and t_1. This motivates the following

Definition 2.14

Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then we call $\mathfrak{L}(P)$ controllable if

$$\mathcal{Z}(P) = \emptyset$$

Corollary 2.15 (Hautus-Test)

Let $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$ and set $P(\lambda) := [\lambda I - A, -B] \in \mathbb{C}[\lambda]^{n,n+m}$.
The (A,B) is controllable if and only if P is left prime.
By Theorem 2.13 this is also equivalent to the controllability of $\mathfrak{L}(P)$ (since $\operatorname{rank}_{\mathbb{C}(\lambda)} P = n$, see Homework).

Proof Combining Theorem 2.6 and Theorem 2.2 ii) the tupel (A,B) is controllable if and only if every solution $z \in \mathcal{C}^n_\infty$ of

$$\begin{cases} \dot{z}(t) = -A^* z(t) \\ 0 = B^* z(t) \end{cases}$$

satisfies $z = 0$. This can also be expressed as

$$\mathfrak{L}\left(\begin{bmatrix} -\lambda I - A^* \\ -B^* \end{bmatrix}\right) = \{0\} \overset{\text{(Homework)}}{\iff} \tilde{P}(\lambda) := \begin{bmatrix} -\lambda I - A^* \\ -B^* \end{bmatrix}$$

is right prime

$$\overset{\text{Homework}}{\iff} P(\lambda) = [\lambda I - A, -B] \text{ is left prime}$$

$\blacksquare$

Controllability and the Lyapunov Equation

In this section we will denote the spectrum of a matrix $A \in \mathbb{C}^{n,n}$ by $\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is eigenvalue of } A\}$

$$\overset{\text{Homework}}{=} \mathcal{Z}(\lambda I - A).$$

Lemma 2.16 Let $W = W^* \in \mathbb{C}^{n,n}$, $A \in \mathbb{C}^{n,n}$ be such that $\sigma(A) \subseteq \mathbb{C}_- := \{z \in \mathbb{C} \mid Re(z) < 0\}$. Then the Lyapunov-equation $\quad A^* X + X A = W$

(which is a matrix equation in the unknown $X \in \mathbb{C}^{n,n}$)

has a unique solution which is given by

$X := -\int_0^\infty e^{At} W e^{A^* t} \, dt$ and is thus Hermitian: $X = X^*$

Proof: We do not show the uniquness here. To show that X solves the Lyapunov equation, define $Z : [0, \infty) \to \mathbb{C}^{n,n}$ by $Z(t) := e^{At} W e^{A^* t}$. Then we have

$$\dot{Z}(t) = A e^{At} W e^{A^* t} + \underbrace{e^{At} W A^* e^{A^* t}}_{e^{At} W e^{A^* t} A^*}$$

$$= A Z(t) + Z(t) A^* \quad \text{and} \quad Z(0) = e^{A \cdot 0} W e^{A^* 0} = W$$

Since $\sigma(A) \subseteq \mathbb{C}_-$ and thus also $\sigma(A^*) \subseteq \mathbb{C}_-$ we have

$$Z_\infty := \lim_{t \to \infty} Z(t) = \underbrace{\left(\lim_{t \to \infty} e^{At}\right)}_{=0} \cdot W \underbrace{\left(\lim_{t \to \infty} e^{A^* t}\right)}_{=0} = 0$$

$$\Rightarrow -W = Z_\infty - Z(0) = \int_0^\infty \dot{Z}(t)\,dt = A \underbrace{\left(\int_0^\infty Z(t)\,dt\right)}_{=X} + \underbrace{\left(\int_0^\infty Z(t)\,dt\right)}_{=X} A^*$$

$\Rightarrow X$ is a solution $\square$

The Lyapunov equation with $\sigma(A) \subseteq \mathbb{C}_-$ can be solved numerically by computation the Schur form of A and then performing certain additional operations ($\rightarrow$ Bartels - Stewart - Algorithm).

<u>Corollary 2.17:</u> Let $A \in \mathbb{C}^{n,n}$ be with $\sigma(A) \subseteq \mathbb{C}_-$ and let $B \in \mathbb{C}^{n,m}$. Then (A, B) is controllable if and only if the unique Hermitian solution $X = X^* \in \mathbb{C}^{n,n}$ of the Lyapunov equation

$$AX + XA^* = -BB^*$$

is positive definite : $X > 0$

<u>Proof:</u> First we note that (A, B) is controllable if and only if $(-A, B)$ is controllable (Homework). Using Theorem 2.6 we find that $(-A, B)$ is controllable if and only if the Gramian of controllability $V(\tau)$ is invertible for $\tau = \infty$, i.e, (since $V(\tau) = V^*(\tau) \geq 0$ anyway) if and only if

$$0 < V(\infty) = \int_0^\infty e^{-(-At)} BB^* e^{-(-A^*t)} dt .$$

This, however, is the unique solution of the specified Lyapunov equation as one can see by the previous Lemma 2.16

$\boxtimes$

Checking controllability numerically

Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$. To check if the system $\mathfrak{B}(\lambda F + G)$ is controllable we show (in the handout) that there exist unitary matrices X, Y such that

$$X(\lambda F + G)Y = \begin{bmatrix} \lambda F_1 + G_1 & * & * \\ 0 & \lambda F_2 + G_2 & * \\ 0 & 0 & \lambda F_3 + G_3 \end{bmatrix}$$

where $*$ denotes first order polynomial matrices of matching size and

- $\lambda F_1 + G_1 \in \mathbb{C}[\lambda]^{p_1, q_1}$ is left prime, F_1 has full row rank (over $\mathbb{C}$)
- $\lambda F_2 + G_2 \in \mathbb{C}[\lambda]^{p_2, p_2}$ and F_2 is invertible (over $\mathbb{C}$)
  $\Rightarrow \lambda F_2 + G_2$ is invertible (over $\mathbb{C}(\lambda)$)
- $\lambda F_3 + G_3 \in \mathbb{C}[\lambda]^{p_3, q_3}$ is right prime

This implies that

$$\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) \overset{!}{=} \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F_1 + G_1)}^{= p_1} + \overbrace{\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F_2 + G_2)}^{= p_2}$$
$$+ \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F_3 + G_3)}_{= q_3}$$
$$= p_1 + p_2 + q_3$$

where as for fixed $\lambda_0 \in \mathbb{C}$ we have

$$\operatorname{rank}(\lambda_0 F + G) \overset{!}{=} \operatorname{rank}(\lambda_0 F_1 + G_1) + \operatorname{rank}(\lambda_0 F_2 + G_2) + \operatorname{rank}(\lambda_0 F_3 + G_3)$$
$$= p_1 + q_3 + \operatorname{rank}(\lambda_0 F_2 + G_2)$$

Using Lemma 1.9. this implies that $\mathfrak{B}(\lambda F + G)$ is controllable if and only if

$$p_2 = \operatorname{rank}(\lambda_0 F_2 + g_2) \qquad \forall \lambda_0 \in \mathbb{C}$$

$$= \operatorname{rank}\Big(\underbrace{F_2}_{\text{invertible}}\big(\lambda_0 I - \underbrace{(-F_2^{-1} g_2)}_{=:\,A \in \mathbb{C}^{p_2,p_2}}\big)\Big) = \operatorname{rank}(\lambda_0 I - A)$$

Since for every eigenvalue λ_0 of A we see with the Jordan form that $\operatorname{rank}(\lambda_0 I - A) < p_2$

we conclude that $\mathfrak{L}(\lambda F + g)$ is controllable if and only if $p_2 = 0$

Stabilizability

One can think of stabilizability as asymptotic controllability.

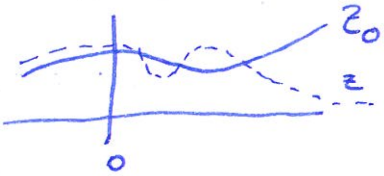
Definition: Let $A \in \mathcal{C}_\infty^{n,n}$ and $B \in \mathcal{C}_\infty^{n,m}$. Then the system

(LTVS) $\qquad \dot{x}(t) = A(t)x(t) + B(t)u(t)$,

is stabilizable from $t_0 \in \mathbb{R}$ if for all $x_0 \in \mathbb{C}^n$ there exists a $u \in \mathcal{C}_\infty^m$ such that the unique solution of (LTVS) with $x(t_0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = 0$

This Definition will not be used in the following.

Definition 2.18: Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then $\mathfrak{L}(P)$ is called stabilizable, if for every $z_0 \in \mathfrak{L}(P)$ there exists a $z \in \mathfrak{L}(P)$ such that $z_0(t) = z(t)$ for all $t \le 0$ and $\lim_{t\to\infty} z^{(k)}(t) = 0$ for all derivatives $k \in \mathbb{N}_0$.



Theorem 2.19: Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then the following are equivalent:

1) $\mathfrak{L}(P)$ is stabilizable.
2) $\mathcal{Z}(P) \subseteq \mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$,
   i.e., all zeros are in the open left half plane.
3) In the Kronecker canonical form of the canonical linearization all blocks of type
   $J_{\ell_j} = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_j \end{bmatrix}$ fullfill $\mathrm{Re}(\lambda_j) < 0$

Proof: 2) $\Leftrightarrow$ 3) handout about the "Kronecker canonical form", Lemma 4.

1) $\Leftrightarrow$ 2) similar to theorem 2.13., 1) $\Rightarrow$ 2).
Not here. Also compare [Polderman, Willems, Thm 5.2.30] $\square$

Observability of LTVS

Definition 2.20

Let $A \in \mathcal{C}_\infty^{n,n}$, $C \in \mathcal{C}_\infty^{p,n}$ and $t_0 < t_1$. Then the time dependent system

$$(*)\ \mathfrak{L} := \left\{ (y,x) \in \mathcal{C}_\infty^{p+n} \;\middle|\; \begin{array}{l} \dot{x}(t) = A(t)x(t) \\ y(t) = C(t)x(t) \end{array} \right\}$$

is called *observable in $[t_0, t_1]$* if for any two trajectories $(y_1,x_1), (y_2,x_2) \in \mathfrak{L}$ we have

$$\left[y_1(t) = y_2(t) \quad \forall t \in [t_0,t_1] \right] \Rightarrow \left[x_1(t) = x_2(t) \quad \forall t \in [t_0,t_1] \right]$$