Reconstructability

One can think of reconstructability as asymptotic observability.

Definition: Let $A \in \mathcal{C}_\infty^{n,n}$ and $C \in \mathcal{C}_\infty^{p,n}$. Then the system
$$\mathcal{L}e := \left\{ (y,x) \in \mathcal{C}_\infty^{p+n} \;\middle|\; \begin{array}{l} \dot{x}(t) = A(t)x(t) \\ y(t) = C(t)x(t) \end{array} \right\}$$
is called reconstructable from $t_0 \in \mathbb{R}$ if for any two trajectories $(y_1, x_1), (y_2, x_2) \in \mathcal{L}e$ we have
$$\left[y_1(t) = y_2(t) \;\; \forall t \in [t_0, \infty) \right] \Rightarrow \left[\lim_{t \to \infty} (x_1(t) - x_2(t)) = 0 \right].$$
This notion will not be used in the following.

Definition 2.32:

Let $R \in \mathbb{C}[\lambda]^{p,q}$, $M \in \mathbb{C}[\lambda]^{p,r}$. For the latent variable description $\mathcal{L}e_f := \left\{ (z, \ell) \in \mathcal{C}_\infty^{q+r} \;\middle|\; R(\tfrac{\partial}{\partial t}) z = M(\tfrac{\partial}{\partial t}) \ell \right\}$

we say that ℓ is reconstructable from z, if for all $(z, \ell_1), (z, \ell_2) \in \mathcal{L}e_f$ we have $\lim_{t \to \infty} (\ell_1(t) - \ell_2(t)) = 0$.

Theorem 2.33:

With the notation of Definition 2.32 the following are equivalent:

1.) ℓ is reconstructable from z.

2.) $\mathcal{Z}(M) \subseteq \mathbb{C}_- \wedge \operatorname{rank}_{\mathbb{C}(\lambda)} M = r$.

3.) For all $\ell \in \mathcal{L}e(M)$ we have $\lim_{t \to \infty} \ell(t) = 0$

Proof: 1) $\Leftrightarrow$ 3) Use linearity of the system (Homework)

3) $\Rightarrow$ 2) If 2) was not true, there would be a $\lambda_0 \in \overline{\mathbb{C}_+} = \{\lambda \in \mathbb{C} \mid Re(\lambda) \geq 0\}$ and a vector $\alpha_0 \in \mathbb{C}^r \setminus \{0\}$ such that $M(\lambda_0)\alpha_0 = 0$.

Setting $\ell(t) := \alpha_0 e^{\lambda_0 t}$ then shows that

$$M(\tfrac{\partial}{\partial t})\ell(t) = M(\lambda_0)\ell(t) = \underbrace{M(\lambda_0)\alpha_0}_{=0} e^{\lambda_0 t} = 0$$

$\Rightarrow \ell \in \mathfrak{L}(M)$ although $\lim_{t\to\infty} \ell(t)$ does not exist.

2) $\Rightarrow$ 3) In this case the Smith form is $M = S\begin{bmatrix} D \\ 0 \end{bmatrix} T$ with $\mathcal{Z}(D) \subseteq \mathbb{C}_-$. Thus $\mathfrak{L}(M)$ $= T^{-1}(\tfrac{d}{dt})\,\mathfrak{L}(\begin{bmatrix} D \\ 0 \end{bmatrix}) = T^{-1}(\tfrac{\partial}{\partial t})\,\mathfrak{L}(D)$ and by assumption the elements of $\mathfrak{L}(D)$ (and all there drivatives) go to zero. $\blacksquare$

Summary $\qquad$ $\operatorname{rank}_{\mathbb{C}(\lambda)}(M) = r$

controllability $\mathcal{Z}(P) = \emptyset$	observability $\mathcal{Z}(M) = \emptyset$
stabilizability $\mathcal{Z}(P) \subseteq \mathbb{C}_-$	reconstructability $\mathcal{Z}(M) \subseteq \mathbb{C}_-$

where $P \in \mathbb{C}[\lambda]^{p,q}$ and $M \in \mathbb{C}[\lambda]^{q,r}$

Chapter 3: Controllers and Observers

Behavioral controllers

A dynamical system $\mathfrak{B} \subseteq \mathcal{C}_\infty^q$ is a set of events (= trajectories) that can occur in reality.

However, we usually want to avoid certain events of reality, namely those which are inconvenient, costly, or dangerous.

In other words, one wants to ensure that only events from a desired subbehavior $\mathfrak{B}_c \subseteq \mathfrak{B}$ can happen, i.e., to restrict reality to our wishes $\mathfrak{B}_c$. $\leftarrow$ controlled behavior

$\boxed{\text{From the behavioral viewpoint control is restriction}}$

In the following we are going to restrict (linear) behaviors by adding (linear) equations.

Definition: Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then we call any $C \in \mathbb{C}[\lambda]^{s,q}$ a controller of $\mathfrak{B}(P)$ and we call

$$\mathfrak{B}_c := \mathfrak{B}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) \subseteq \mathfrak{B}(P)$$

the associated controlled behavior.

Proposition: Let $R \in \mathbb{C}[\lambda]^{r,q}$, $P \in \mathbb{C}[\lambda]^{p,q}$. Then equivalent are:

i) There exists a controller $C \in \mathbb{C}[\lambda]^{s,q}$ such that $\mathfrak{B}(R) = \mathfrak{B}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$

ii) $\mathcal{L}e(R) \subseteq \mathcal{L}e(P)$

Proof: Homework. For ii) $\Rightarrow$ i) Choose $C := R$. ▨

However, usually we only want to add as few equations as possible, i.e., we want to pick a controller $C \in \mathbb{C}[\lambda]^{s,q}$ with minimal s; and not just simply $C = R$.

To clarify this we need following:

Lemma 3.1.: Let $P \in \mathbb{C}[\lambda]^{p,q}$, $R \in \mathbb{C}[\lambda]^{s,q}$.

Then the following statements hold:

1.) We have $\mathcal{L}e(R) \subseteq \mathcal{L}e(P)$ if and only if there exists a $N \in \mathbb{C}[\lambda]^{p,s}$ such that $P = N \cdot R$

2.) Assume $p = s$. Then we have $\mathcal{L}e(R) = \mathcal{L}e(P)$ if and only if there exists a unimodular $N \in \mathbb{C}[\lambda]^{p,p}$ such that
$$P = N \cdot R$$

3.) If $\mathcal{L}e(R) = \mathcal{L}e(P)$ then $\operatorname{rank}_{\mathbb{C}(\lambda)} R = \operatorname{rank}_{\mathbb{C}(\lambda)} P$

Proof: 1) "$\Rightarrow$" Let $R = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$, $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{bmatrix} \in \mathbb{C}[\lambda]^{r,r}$

be the Smith-form. Then we have

$$\mathcal{L}e\left(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{L}e(S^{-1} R T^{-1}) = T\left(\tfrac{d}{dt}\right) \mathcal{L}e(R) \subseteq T\left(\tfrac{d}{dt}\right) \mathcal{L}e(P)$$
$$= \mathcal{L}e(\underbrace{P T^{-1}}_{=: \left[\, \underset{r \text{ cols}}{P_1} \,\vdots\, \underset{q-r \text{ cols}}{P_2} \,\right]}) \qquad (\square)$$

which means that for all $(z_1, z_2) \in \mathcal{C}_\infty^r \times \mathcal{C}_\infty^{q-r}$ with

$D\left(\frac{d}{dt}\right) z_1 + 0 \cdot z_2 = 0$ we have $P_1\left(\frac{d}{dt}\right) z_1 + P_2\left(\frac{d}{dt}\right) z_2 = 0$. $(*)$

Choosing $z_1 = 0$ in (*) this implies that for all $z_2 \in \mathcal{C}_\infty^{q-r}$ we have $P_2(\frac{d}{dt}) z_2 = 0 \Rightarrow \mathfrak{L}(P_2) = \mathcal{C}_\infty^{q-r}$

$\overset{\text{(Homework)}}{\Longrightarrow} P_2 = 0.$

Choosing $z_2 = 0$ in (*) implies that for all $z_1 \in \mathfrak{L}(D)$ we have $P_1(\frac{d}{dt}) z_1 = 0 \Rightarrow \mathfrak{L}(D) \subseteq \mathfrak{L}(P_1)$

$\overset{\text{(Homework)}}{\Longrightarrow} \exists M \in \mathbb{C}[\lambda]^{s,r}$ with $P_1 = M \cdot D$

$$\Rightarrow P = [P_1, P_2] T^{-1} = \underbrace{[MD, 0]}_{\in \mathbb{C}[\lambda]^{p,q}} T^{-1} = \underbrace{[M, 0] S}_{=: N \in \mathbb{C}[\lambda]^{p,s}} \underbrace{S^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T^{-1}}_{\in \mathbb{C}[\lambda]^{s,q}}$$

$= N \cdot R.$

"$\Leftarrow$" Excercise.

2) Withe the notation and derivations of part 1.) we find

as in (□) $\mathfrak{L}\left(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathfrak{L}\left(\begin{bmatrix} MD, 0 \end{bmatrix}\right) \Rightarrow \mathfrak{L}(D) = \mathfrak{L}(M \cdot D).$

$\overset{\text{(Homework)}}{\Longrightarrow} \mathfrak{L}(M) = \{0\} \Rightarrow M$ is right prime

Using Theorem 1.13. there exists a M' such that $[M, M']$ is unimodular.

$$\Rightarrow P = [MD, 0] T^{-1} = [M, M'] \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

$$= \underbrace{[M, M'] S}_{=: N} S^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T^{-1} = N \cdot R$$

"$\Leftarrow$" Excercise

3.) Append zeros rows to R or P such that

$$\tilde{P} := \begin{bmatrix} P \\ 0 \end{bmatrix}, \tilde{R} := \begin{bmatrix} R \\ 0 \end{bmatrix} \in \mathbb{C}[\lambda]^{\tilde{p},q}$$ have the same

number of rows. Then

$$\mathfrak{L}(\tilde{P}) = \mathfrak{L}(P) = \mathfrak{L}(R) = \mathfrak{L}(\tilde{R}).$$

$$\Rightarrow \operatorname{rank}_{\mathbb{C}(\lambda)} P = \operatorname{rank}_{\mathbb{C}(\lambda)} \tilde{P} \underset{\text{use 2.)}}{=} \operatorname{rank}_{\mathbb{C}(\lambda)} \tilde{N}\tilde{R} \overset{\tilde{N}\text{ unimodular}}{=} \operatorname{rank}_{\mathbb{C}(\lambda)} \tilde{R}$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} R.$$

$\blacksquare$

Let $P \in \mathbb{C}[\lambda]^{p,q}$, $R \in \mathbb{C}[\lambda]^{s,q}$ with $\mathfrak{L}(R) \subseteq \mathfrak{L}(P)$
and let $C \in \mathbb{C}[\lambda]^{c,q}$ be with

$$\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = \mathfrak{L}(R)$$

$$\text{Then } \operatorname{rank}_{\mathbb{C}(\lambda)} R \overset{\text{Lemma 3.1.c)}}{=} \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} P \\ C \end{bmatrix} \leq \operatorname{rank}_{\mathbb{C}(\lambda)} P + \operatorname{rank}_{\mathbb{C}(\lambda)} C$$

$$\Rightarrow \operatorname{rank}_{\mathbb{C}(\lambda)} C \geq \operatorname{rank}_{\mathbb{C}(\lambda)} R - \operatorname{rank}_{\mathbb{C}(\lambda)} P =: c_0$$

which means that c_0 is a candidate for the minimal number of rows in C.

<u>Definition:</u> Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then $C \in \mathbb{C}[\lambda]^{c,q}$ is called a <u>regular controller</u> for the system $\mathfrak{L}(P)$ if

$$c = \operatorname{rank}_{\mathbb{C}(\lambda)} C = \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} P \\ C \end{bmatrix} - \operatorname{rank}_{\mathbb{C}(\lambda)} P.$$

Theorem 3.2:

Let $P \in \mathbb{C}[\lambda]^{p,q}$ be such that $\mathfrak{L}(P)$ is controllable. Then for every $R \in \mathbb{C}[\lambda]^{s,q}$ with $\mathfrak{L}(R) \subseteq \mathfrak{L}(P)$ there exists a regular controller $C \in \mathbb{C}[\lambda]^{c,q}$ such that

$$\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = \mathfrak{L}(R)$$

Proof: Since $\mathfrak{L}(P)$ is controllable the Smith form is $P = S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T$ and we have

$$\mathfrak{L}\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right) = T^{-1}\left(\tfrac{d}{dt}\right)\mathfrak{L}(P) \supseteq T^{-1}\left(\tfrac{d}{dt}\right)\mathfrak{L}(R)$$

$$= \mathfrak{L}(\underbrace{[R \cdot T]}_{=:\,[\underset{r \text{ cols}}{R_1} \,\vdots\, \underset{q-r \text{ cols}}{R_2}]})$$

Let $z_1 \in \mathfrak{L}(R_1)$. Then $\begin{bmatrix} z_1 \\ 0 \end{bmatrix} \in \mathfrak{L}([R_1, R_2]) \subseteq \mathfrak{L}\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)$

$\Rightarrow z_1 = 0 \Rightarrow \mathfrak{L}(R_1) = \{0\} \Rightarrow R_1$ is right prime.

Using Theorem 1.13. this shows that the Smith form of R_1 is $R_1 = \tilde{S}_1 \begin{bmatrix} I \\ 0 \end{bmatrix} \tilde{T}_1 = \tilde{S}_1 \begin{bmatrix} \tilde{T}_1 \\ 0 \end{bmatrix} = \underbrace{\tilde{S}_1 \begin{bmatrix} \tilde{T}_1 & 0 \\ 0 & I \end{bmatrix}}_{=:\,S_1 \text{ unimodular}} \begin{bmatrix} I \\ 0 \end{bmatrix}$

$$\Rightarrow S_1^{-1}[R_1\ R_2] =: \left[\begin{array}{c|c} I & X_1 \\ 0 & X_2 \end{array}\right]$$

$$\Rightarrow \mathfrak{L}\left(\begin{bmatrix} I & X_1 \\ 0 & X_2 \end{bmatrix}\right) = \mathfrak{L}([R_1, R_2]) \subseteq \mathfrak{L}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right)$$

Let $\tilde{z}_2 \in \mathfrak{L}(X_2)$ be arbitrary. Set $\tilde{z}_1 := -X\left(\tfrac{d}{dt}\right)\tilde{z}_2$ so that

$$\begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \in \mathfrak{L}e\left(\begin{bmatrix} I & X_1 \\ 0 & X_2 \end{bmatrix}\right) \subseteq \mathfrak{L}e\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) \Rightarrow \tilde{z}_1 = 0 \Rightarrow X_1\left(\tfrac{d}{dt}\right)\tilde{z}_2 = 0$$

$$\Rightarrow \mathfrak{L}e(X_2) \subseteq \mathfrak{L}e(X_1) \overset{\text{Lemma 3.1}}{\Longleftrightarrow} \exists N \text{ such that } N X_2 = X_1 .$$

$$\Rightarrow \mathfrak{L}e\left(\begin{bmatrix} I & X_1 \\ 0 & X_2 \end{bmatrix}\right) = \mathfrak{L}e\left(\begin{bmatrix} I & N X_2 \\ 0 & X_2 \end{bmatrix}\right) = \mathfrak{L}e\left(\underbrace{\begin{bmatrix} I & N \\ 0 & I \end{bmatrix}}_{\text{unimodular}} \begin{bmatrix} I & 0 \\ 0 & X_2 \end{bmatrix}\right)$$

$$= \mathfrak{L}e\left(\begin{bmatrix} I & 0 \\ 0 & X_2 \end{bmatrix}\right)$$

Finally choosing U unimodular such that

$U X_2 = \begin{bmatrix} X_3 \\ 0 \end{bmatrix}$, where X_3 has full row rank and

setting $C := [0, X_3]T^{-1}$ we see that C has full

row rank and
$$\begin{bmatrix} P \\ C \end{bmatrix} = \begin{bmatrix} S^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \\ [0, X_3]T^{-1} \end{bmatrix} = \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & X_3 \end{bmatrix} T^{-1}$$

which implies $\operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} P \\ C \end{bmatrix} = r + \operatorname{rank}_{\mathbb{C}(\lambda)} X_3$

$= \operatorname{rank}_{\mathbb{C}(\lambda)} P + \operatorname{rank}_{\mathbb{C}(\lambda)} C \Rightarrow$ controller is regular

and $\mathfrak{L}e\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = T^{-1}\left(\tfrac{d}{dt}\right) \mathfrak{L}e\left(\begin{bmatrix} I & 0 \\ 0 & X_3 \\ 0 & 0 \end{bmatrix}\right) = T^{-1}\left(\tfrac{d}{dt}\right) \mathfrak{L}e\left(\begin{bmatrix} I & 0 \\ 0 & X_2 \end{bmatrix}\right)$

$= \underset{\text{see above}}{\ldots} = T^{-1}\left(\tfrac{d}{dt}\right) \mathfrak{L}e\left(\begin{bmatrix} I & X_1 \\ 0 & X_2 \end{bmatrix}\right) = T^{-1}\left(\tfrac{d}{dt}\right) \mathfrak{L}e([R_1, R_2])$

$= \mathfrak{L}e(R)$ ∎

Stabilization

In this section we are going to "design" controllers which make a system stable.

<u>Definition:</u> Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then $\mathfrak{L}e(P)$ is called <u>stable</u> if for all $z \in \mathfrak{L}e(P)$ we have $\lim\limits_{t\to\infty} z(t) = 0$.

<u>Lemma 3.3.:</u>

Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then $\mathfrak{L}e(P)$ is stable if and only if

- $\mathfrak{L}e(P)$ is automonous $\left(\overset{\text{Thm. 1.19.}}{\Longleftrightarrow} \operatorname{rank}_{\mathbb{C}(\lambda)} P = q\right)$
- $\mathcal{Z}(P) \subseteq \mathbb{C}_-$

<u>Proof:</u> „$\Rightarrow$" If $\mathfrak{L}e(P)$ was not autonomous then there would be free components u (compare Theorem 1.19) which one can then choose such that $\lim\limits_{t\to\infty} \|u(t)\| = \infty$ ↯

If $\mathcal{Z}(P) \subseteq \mathbb{C}_-$ would not hold, then would be a $\hat{\lambda} \in \mathcal{Z}(P)$, $Re(\hat{\lambda}) \geq 0$. $\overset{\text{Lemma 1.19}}{\Longrightarrow} \operatorname{rank}(P(\hat{\lambda})) < q \Rightarrow \exists\, \hat{\alpha} \neq 0$ with $P(\hat{\lambda})\hat{\alpha} = 0$.

Setting $\hat{z}(t) := \hat{\alpha} e^{\hat{\lambda} t}$ shows that $P(\frac{d}{dt})\hat{z}(t) = \underbrace{P(\hat{\lambda})\hat{\alpha}}_{=0} e^{\hat{\lambda} t} = 0$ although $\hat{z}(t)$ does not converge to zero. ↯

„$\Leftarrow$" Same as in the proof of Theorem 2.33. (2.) $\Rightarrow$ 3.))

Consider the Smith form and explicitly give all solutions of the system in the Smith form. ▨

Theorem 3.4:

Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $\Lambda \subseteq \mathbb{C}$ be a finite set (i.e, $|\Lambda| < \infty$) with $\mathcal{Z}(P) \subseteq \Lambda$. Then there exists a regular controller C such that

- $\mathcal{Z}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = \Lambda$
- $\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$ is autonomous, and
- $\mathcal{Z}(C) = \Lambda \setminus \mathcal{Z}(P)$

Proof: Let the Smith form be $P = S \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T$, $D \in \mathbb{C}[\lambda]^{r,r}$. Choose some invertible $D_C \in \mathbb{C}[\lambda]^{q-r,q-r}$ with $\mathcal{Z}(D_C) = \Lambda / \mathcal{Z}(P)$, for example if $\Lambda / \mathcal{Z}(P) =: \{\mu_1, \dots, \mu_k\}$ choose $D_C(\lambda) = \begin{bmatrix} 1 & & \\ & \ddots 1 & \\ & & \chi(\lambda) \end{bmatrix}$, with $\chi(\lambda) := (\lambda - \mu_1) \cdot \ldots \cdot (\lambda - \mu_k)$.

Similar to the proof of Theorem 3.3 one shows that then $C := [0, D_C]T$, is a regular controller.

Furthermore, $\begin{bmatrix} P \\ C \end{bmatrix} = \left[\begin{array}{c|c} S & \\ \hline & I \end{array}\right] \begin{bmatrix} D & 0 \\ 0 & 0 \\ \hline 0 & D_C \end{bmatrix} T$ has full column rank.

By Theorem 1.19 this implies that $\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$ is autonomous and $\mathcal{Z}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & D_C \end{bmatrix}\right) = \mathcal{Z}(D) \cup \mathcal{Z}(D_C) = \Lambda$

$\blacksquare$

<u>Definition:</u> Let $P \in \mathbb{C}[\lambda]^{p,q}$. We call $C \in \mathbb{C}[\lambda]^{c,q}$ a stabilizing controller for $\mathfrak{L}(P)$ if $\mathfrak{L}\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$ is stable.

<u>Corollary 3.5:</u>

Let $P \in \mathbb{C}[\lambda]^{p,q}$ be such that $\mathfrak{L}(P)$ is stabilizable. Then there exists a regular, stabilizing, and left prime controller of $\mathfrak{L}(P)$.

<u>Proof:</u> Homework; use Theorem 2.19, Lemma 3.3, Theorem 3.4 and also Theorem 1.13. $\blacksquare$