The algebraic Riccati equation

In this section we interpret the results of the previous section for state-space systems.

Therefor, let A, $Q = Q^* \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, $R = R^* \in \mathbb{C}^{m,m}$ and consider the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x \in \mathcal{C}^n_\infty$ is the state and $u \in \mathcal{C}^m_\infty$ is the input, and assume that we measure the cost of any trajectory at time $t \in \mathbb{R}$ via

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Then we can set $F := [I, 0]$, $G := [-A, -B]$, and $H := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ and use $z = \begin{bmatrix} x \\ u \end{bmatrix}$ to rewrite the system as $\mathcal{L}_e(\lambda F + G) = \mathcal{L}_e([\lambda I - A, -B])$ and the cost at $t \in \mathbb{R}$ as $z^*(t) H z(t)$.

Clearly, F has full row rank. Also, by Theorem 2.15., we know that $\mathcal{L}_e(\lambda F + G)$ is controllable if the tupel (A, B) is controllable. If we assume this and if we can make sure that

$$0 < H = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$$

(for example by choosing $Q = I_n$, $R = I_m$, $S = 0$)

then the Assumptions 4.8 are fulfilled and we can apply Theorem 4.10 to obtain a $Z \in \mathbb{C}^{n,n+m}$ and a $C \in \mathbb{C}^{(n+m)-n,\, m+n} = \mathbb{C}^{m,n+m}$

which has full row rank $(C) = m$ (since C is a regular controller) such that

$$\left[\begin{array}{c|c} 0 & \lambda F + G \\ \hline -\lambda F^* + G^* & C^* C \end{array}\right] = \begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}$$

$$\overset{S.11.T.3}{=} \dots = \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & \lambda(F^* Z - Z^* F) + H + G^* Z + Z^* G \end{bmatrix}$$

which is equivalent (S.11.T.3) to

$$F^* Z = Z^* F \quad \text{and} \quad C^* C = H + G^* Z + Z^* G.$$

This implies that $0 \le H + G^* Z + Z^* G$ and

(✱) $\operatorname{rank}(H + G^* Z + Z^* G) = \operatorname{rank}(C^* C) = \operatorname{rank}(C) = m$.

Partitioning $Z =: [\underbrace{X}_{n \text{ cols}} \,\vdots\, \underbrace{Y}_{m \text{ cols}}] \in \mathbb{C}^{n,n+m}$ this means that

$$F^* Z = \begin{bmatrix} I \\ 0 \end{bmatrix} [X, Y] = \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}$$

$$\Vert$$

$$Z^* F = \begin{bmatrix} X^* \\ Y^* \end{bmatrix} [I, 0] = \begin{bmatrix} X^* & 0 \\ Y^* & 0 \end{bmatrix}$$

$\Rightarrow X^* = X$ is Hermitian and $Y = 0$

and then (by (✱)) that m is the rank of

$$0 \le H + G^* Z + Z^* G = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} -A^* \\ -B^* \end{bmatrix} [X, 0] + \begin{bmatrix} X^* \\ 0 \end{bmatrix} [-A, -B]$$

$$= \left[\begin{array}{c|c} Q - A^*X - X^*A & S - X^*B \\ \hline S^* - B^*X & R \end{array}\right].$$

Thus m is also the rank of

$$\begin{bmatrix} I & -(S-X^*B)R^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q-A^*X-X^*A & S-X^*B \\ S^*-B^*X & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}(S^*-B^*X) & I \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} I & -(S-X^*B)R^{-1} \\ 0 & I \end{bmatrix}} \left[\begin{array}{c|c} Q-A^*X-X^*A-(S-X^*B)R^{-1}(S^*-B^*X) & \cancel{(S-X^*B)} \\ \hline 0 & R \end{array}\right].$$

Since $0 < R \in \mathbb{C}^{m,m}$ is invertible this means that $X = X^*$ solves the *algebraic Riccati equation* (associated with the initially mentioned system and cost functional)

$$0 = Q - A^*X - X^*A - (S-X^*B)R^{-1}(S^*-B^*X). \quad \text{(ARE)}$$

Furthermore, let $R = L^*L$ be a Cholesky factorization of R.

Then $C := \left[L^{-*}(S^*-B^*X), L\right]$

satisfies

$$C^*C = \begin{bmatrix} (S-X^*B)L^{-1} \\ L^* \end{bmatrix} \left[L^{-*}(S^*-B^*X), L\right]$$

$$= \left[\begin{array}{c|c} (S-X^*B)\underbrace{L^{-1}L^{-*}}_{=R^{-1}}(S^*-B^*X) & S-X^*B \\ \hline S^*-B^*X & \underbrace{L^*L}_{=R} \end{array}\right]$$

$$\overset{(ARE)}{=} \left[\begin{array}{c|c} Q-A^*X-X^*A & S-X^*B \\ \hline S^*-B^*X & R \end{array}\right] = H + G^*Z + Z^*G.$$

Thus, by Theorem 4.10 we know that

$$\mathbb{C}_- \supseteq \mathcal{Z}\left(\begin{bmatrix}\lambda F + S\\ C\end{bmatrix}\right) = \mathcal{Z}\left(\left[\begin{array}{c|c}\lambda I - A & -B\\ \hline L^*(S^*-B^*X) & L\end{array}\right]\right)$$

$$= \mathcal{Z}\left(\begin{bmatrix}I & 0\\ 0 & L^*\end{bmatrix}\begin{bmatrix} & \\ & \cdots \end{bmatrix}\right)$$

$$= \mathcal{Z}\left(\begin{bmatrix}\lambda I - A & -B\\ S^*-B^*X & R\end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix}I & BR^{-1}\\ 0 & I\end{bmatrix}\begin{bmatrix} & \\ & \cdots \end{bmatrix}\right)$$

$$= \mathcal{Z}\left(\left[\begin{array}{c|c}\lambda I - A + BR^{-1}(S^*-B^*X) & 0\\ \hline S^*-B^*X & R\end{array}\right]\right)$$

$$= \mathcal{Z}\left(\lambda I - (A - BR^{-1}(S^*-B^*X))\right) = \sigma\left(A - BR^{-1}(S^*-B^*X)\right).$$

We have proven the following:

<u>Corollary 4.11:</u>

Let $(A,B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ be controllable and let $Q = Q^* \in \mathbb{C}^{n,n}$, $S \in \mathbb{C}^{n,m}$ and $R = R^* \in \mathbb{C}^{m,m}$ be such that

$$0 < \begin{bmatrix}Q & S\\ S^* & R\end{bmatrix}$$

then the algebraic Riccati equation

$$0 = Q - A^*X - X^*A - (S - X^*B)R^{-1}(S^* - BX^*)$$

has a Hermitian solution $X = X^* \in \mathbb{C}^{n,n}$ such that

$$\sigma(A - \underbrace{BR^{-1}(S^*-B^*X)}_{=:G}) \subseteq \mathbb{C}_-.$$

This means that G solves the problem of stabilization via state-feedback from Chapter III.

Proof: see above $\square$

Theorem 4.10 also implies that the closed-loop system obtained via state-feedback

$$u(t) = -G\,x(t)$$

given by

$$\dot{x}(t) = A\,x(t) + B\,u(t) = (A - BG)\,x(t) \quad \text{is stable}$$

and optimal in the sense of (LQ).

Remark: The solution of the algebraic Riccati equation can be computed in MATLAB via the function

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>> care,
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which stands for "continuous-time algebraic Riccati equation". The algorithm computes the eigenvalues in $\mathbb{C}_-$ of a Hamiltonian matrix.

Non-linear state-space control problems

Assume that $F : \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}^k \to \mathbb{C}^n$ is some globally Lipschitz continuous function. Then (by the Theorem of Picard-Lindelöf) the initial value problem

$$(NLP) \quad \begin{cases} \dot{x}(t) = F(x(t), u(t), w(t)) \\ x(0) = x_0 \end{cases}$$

where
- $x \in \mathcal{L}_\infty^n$ is the state,
- $u \in \mathcal{L}_\infty^m$ is the input/control
- $w \in \mathcal{L}_\infty^k$ is some external disturbance

has a unique solution for every $x_0 \in \mathbb{C}^n$, $u \in \mathcal{L}_\infty^m$, $w \in \mathcal{L}_\infty^k$.

If we furthermore, assume that F is sufficiently often continuously differentiable we can perform Taylor expansion of F around some distinguished point $\hat{z} := (\hat{x}, \hat{u}, \hat{w}) \in \mathbb{C}^{n+m+k}$ to obtain that for all $z := (x, u, w) \in \mathbb{C}^{n+m+k}$ we have

$$\begin{aligned} F(x,u,w) = F(z) &= F(\hat{z}) + DF(z)(z - \hat{z}) + \mathcal{O}(\|z - \hat{z}\|^2) \\ &\approx F(\hat{z}) + DF(\hat{z})(z - \hat{z}) \\ &= F(\hat{x}, \hat{u}, \hat{w}) + \left[\frac{\partial}{\partial x} F(\hat{z}), \frac{\partial}{\partial u} F(\hat{z}), \frac{\partial}{\partial w} F(\hat{z})\right] \begin{bmatrix} x - \hat{x} \\ u - \hat{u} \\ w - \hat{w} \end{bmatrix}. \end{aligned}$$

With the definitions $A := \frac{\partial}{\partial x} F(\hat{z}) \in \mathbb{C}^{n,n}$

$B := \frac{\partial}{\partial u} F(\hat{z}) \in \mathbb{C}^{n,m}$

$W := \frac{\partial}{\partial w} F(\hat{z}) \in \mathbb{C}^{n,k}$

this means that

$$\dot{x}(t) = F(x(t), u(t), w(t)) \approx F(\hat{x}, \hat{u}, \hat{w}) + [A, B, W] \begin{bmatrix} x(t) - \hat{x} \\ u(t) - \hat{u} \\ w(t) - \hat{w} \end{bmatrix}$$

$$= F(\hat{x}, \hat{u}, \hat{w}) + A(x(t) - \hat{x}) + B(u(t) - \hat{u}) + W(w(t) - \hat{w})$$

could be a good approximation to the original system (NLP) in a neighbourhood of $(\hat{x}, \hat{u}, \hat{w})$.

If, furthermore, $\hat{u} = 0, \hat{w} = 0$ and $(\hat{x}, \hat{u}, \hat{w}) = (\hat{x}, 0, 0)$ is a steady state point of F

(i.e., $F(\hat{x}, 0, 0) = 0$, so that the constant function $x(t) :\equiv \hat{x}$ is a solution of (NLP) : $\dot{x}(t) = \frac{d}{dt} \hat{x} = 0 = F(\hat{x}, 0, 0)$
$= F(x(t), 0, 0)$)

we even have

$$\dot{x}(t) \approx \underbrace{F(\hat{x}, \hat{u}, \hat{w})}_{=0} + A(x(t) - \hat{x}) + B(u(t) - \hat{u}) + W(w(t) - \hat{w})$$

$$= A(x(t) - \hat{x}) + Bu(t) + Ww(t).$$

By setting $\tilde{x}(t) := (x(t) - \hat{x})$ (the deviation from the steady state) this implies that the linearized) time-invariant state-space system

$$(CIN) \begin{cases} \dot{\tilde{x}}(t) = \frac{d}{dt}(x(t) - \hat{x}) = \dot{x}(t) \approx A(x(t) - \hat{x}) + Bu(t) + Ww(t) \\ \qquad = A\tilde{x}(t) + Bu(t) + Ww(t) \end{cases}$$

could be a good approximation for (NLP).

If $(A, B) = \left(\frac{\partial}{\partial x} F(\hat{z}), \frac{\partial}{\partial u} F(\hat{z})\right)$ is controllable, we can use Corollary 4.11, to construct a $G \in \mathbb{C}^{m,n}$ such that

$$\sigma(A - BG) \subset \mathbb{C}_-.$$

This means if we choose

$$u(t) := -G\tilde{x}(t) = -G(x(t) - \hat{x}) \qquad (*)$$

in (LIN) the solution $\tilde{x}$ of the closed loop linearized system

$$\dot{\tilde{x}}(t) = (A - BG)\tilde{x}(t) + Ww(t)$$

always converges to zero $\lim_{t\to\infty} \tilde{x}(t) = 0$ if the external disturbance $w(t)$ is zero (or at least is "small enough").

$$\Rightarrow \lim_{t\to\infty} x(t) = \hat{x}.$$

We can also use this feedback $(*)$ for (NLP) to obtain the system

$$(CNLP) \begin{cases} \dot{x}(t) = F(x(t), -G(x(t) - \hat{x}), w(t)) \\ x(0) = x_0 \end{cases}$$

If we are lucky, we also have that the solutions of (CNLP) satisfy $\lim_{t\to\infty} x(t) = \hat{x}$, if the external disturbances are small enough.