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Control Theory

1. Exercise

Exercise 1.1: (Inverted Pendulum)
Consider the mathematical model of a controlled inverted pendulum which, after linearization, is given by a second order differential equation

$$\ddot{\varphi}(t) - \varphi(t) = u(t).$$

Here, $\varphi(t) = \theta(t) - \pi$ is the angular deviation of the pendulum from the (upright) equilibrium at time $t \geq 0$ and $u(t)$ is the applied torque.

(a) Show that for proportional feedback $u(t) = -\alpha\varphi(t)$ with $\alpha < 1$ it holds that: if the initial values satisfy $\dot{\varphi}(0) = -\varphi(0)\sqrt{1-\alpha}$, then $\lim_{t\to\infty} \varphi(t) = 0$.

(b) Let $\alpha \in \mathbb{R}$ be fixed. Consider the (energy-)function

$$V(x, y) := \cos x - 1 + \frac{1}{2}(\alpha x^2 + y^2).$$

Show that $V(\varphi(t), \dot{\varphi}(t)) = \mathit{const.}$ along solutions of the nonlinear pendulum equation with proportional feedback

$$\ddot{\varphi}(t) - \sin\varphi(t) + \alpha\varphi(t) = 0. \tag{1}$$

Furthermore, show that there exist initial conditions $\varphi(0) = \varepsilon$, $\dot{\varphi}(0) = 0$, such that the solution of (1) for arbitrary small ε satisfies **not** $\lim_{t\to\infty} \varphi(t) = 0$, $\lim_{t\to\infty} \dot{\varphi}(t) = 0$.

Hint: use that $x = 0$ is an isolated root of $V(x, 0)$. (This follows, since V is analytic. Otherwise it would hold that $V \equiv 0$.)

Exercise 1.2: (Control of a parabolic antenna)
The control problem of a parabolic antenna that is continually directed to a satellite or spacecraft leads to the following simplified differential equations:

$$\begin{aligned} \dot{\varphi}(t) &= \omega(t), \\ j\dot{\omega}(t) &= -r\omega(t) + ku(t), \end{aligned} \tag{2}$$

whereby φ is the rotation angle, ω is the rotational velocity, j is the moment of inertia of the engine that causes the rotation, r is a coefficient of friction and k is an amplification factor. The controlling function is given by the input voltage $u(t)$ that drives the engine. Determine the solution of (2) with initial values $\varphi(0) = 0$ and $\omega(0) = \omega_0$ for the following control functions:

(a) $u(t) = 0$ (free system),

(b) $u(t) = \alpha\omega(t), \quad \alpha \in \mathbb{R}$,

(c) $u(t) = e^{-t}$.

How can $u(t)$ be chosen, such that $\varphi(1) = \pi$?

Exercise 1.3: (Fundamental solution)
Show the following properties of the fundamental solution matrix $\Phi(t,s)$ of $\dot{x}(t) = A(t)x(t)$:

(a) $\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s)$ for all $t, s, \tau \in \mathbb{R}$.

(b) $\Phi(t,s)$ is invertible for all $t, s \in \mathbb{R}$ and it holds that $\Phi(t,s)^{-1} = \Phi(s,t)$.

(c) $\frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t,s)A(s)$.