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Control Theory

1. Exercise

(Discussion on May 5, 2014)

Exercise 1.1: (Control of a parabolic antenna)
The control problem of a parabolic antenna that is continually directed to a satellite or spacecraft leads to the following simplified differential equations:

$$\begin{aligned} \dot{\varphi}(t) &= \omega(t), \\ j\dot{\omega}(t) &= -r\omega(t) + ku(t), \end{aligned} \tag{1}$$

whereby φ is the rotation angle, ω is the rotational velocity, j is the moment of inertia of the engine that causes the rotation, r is a coefficient of friction and k is an amplification factor. The controlling function is given by the input voltage $u(t)$ that drives the engine. Determine the solution of (1) with initial values $\varphi(0) = 0$ and $\omega(0) = \omega_0$ for the following control functions:

(a) $u(t) = 0$ (free system),

(b) $u(t) = \alpha\omega(t), \quad \alpha \in \mathbb{R}$,

(c) $u(t) = e^{-t}$.

How can $u(t)$ be chosen, such that $\varphi(1) = \pi$?

Exercise 1.2: (Laplace transformation)
For a real-valued function f on $[0, \infty)$

$$\mathcal{L}(f) := \hat{f}(s) := \int_0^\infty e^{-st} f(t) dt$$

is called the *Laplace transformation* of f. Show that for real-valued functions f and g on $[0, \infty)$ and $a, b \in \mathbb{R}$ it holds that:

(a) $\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$

(b) $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$

(c) $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}(f(t))$

(d) $\mathcal{L}(f^{(n)}(t)) = s^n\mathcal{L}(f(t)) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$

(e) $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ for $Re(s) > a$

(f) $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ for $Re(s) > 0$.

Exercise 1.3: (Fundamental solution)

1. Show the following properties of the fundamental solution matrix $\Phi(t, s)$ of $\dot{x}(t) = A(t)x(t)$:
   (a) $\Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s)$ for all $t, s, \tau \in \mathbb{R}$.
   (b) $\Phi(t, s)$ is invertible for all $t, s \in \mathbb{R}$ and it holds that $\Phi(t, s)^{-1} = \Phi(s, t)$.
   (c) $\frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t, s)A(s)$.
2. Consider the adjoint equation
   $$\dot{z}(t) = -A(t)^T z(t) \tag{2}$$
   Show that for the fundamental solution $\Psi(t, s)$ of (2) it holds that: $\Psi(t, s) = \Phi(s, t)^T$.