Technische Universität Berlin
Institut für Mathematik
Lena Scholz

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Control Theory

2. Exercise
(Discussion on May 12, 2014)

Exercise 2.1: (Controllability Gramian)
Show that the following statements hold:

(a) The (t_0, t_1)-controllability Gramian $W(t_0, t_1)$ of an LTV system is positive definite if and only if

$$\hat{W}(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, t)B(t)B(t)^T \Phi(t_1, t)^T dt$$

is positive definite.

(b) An asymptotically stable (for $u = 0$) LTI system is controllable if and only if the controllability Gramian

$$W := \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

is positive definite.

Exercise 2.2: (Hautus test for stabilizable systems)
Show that the following statements are equivalent for an LTI system of the form $\dot{x} = Ax + Bu$:

(a) The LTI system is stabilizable.

(b) If v is an eigenvector of A^T for an eigenvalue λ with $Re(\lambda) \geq 0$ then $v^T B \neq 0$.

(c) $rank([A - \lambda I, B]) = n$ for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq 0$.

(d) Let $A = V \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} V^T$, $B = V \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ be the Kalman-decomposition of (A, B), then $\sigma(A_3) \subset \mathbb{C}^-$.

Exercise 2.3: (Controllability)
Check if the following systems are controllable and stabilizable:

(a) $\dot{x} = x + [0, 0, ..., 0, 1]^T u$

(b) $$\dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$