Technische Universität Berlin Summer term 2014
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Control Theory

5. Exercise

(Discussion on June 23, 2014)

Exercise 5.1: (Ackermann formula)

Let $(A, B) \in \mathbb{R}^{n,n} \times \mathbb{R}^{n,1}$ be controllable with Regelungsnormalform

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}, \quad T^{-1}B = e_n := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and let $\mathcal{L} = \{\mu_1, \dots, \mu_n\}$ be closed under complex conjugation. In the following, $e_i \in \mathbb{R}^n$ denotes the i-th unit vector for $i = 1, \dots, n$ and

$$\Psi(x) := (x - \mu_1) \cdots (x - \mu_n).$$

Show:

(a) The matrix $T^{-1}K(A, B)$ has the property $e_1^T T^{-1} K(A, B) = e_n^T$.

(b) It holds that

$$e_1^T (T^{-1}AT)^k = \begin{cases} e_{k+1}^T & \text{for } k = 0, \dots, n-1, \\ [-\alpha_0, \dots, -\alpha_{n-1}] & \text{for } k = n. \end{cases}$$

(c) Let $\beta_0, ..., \beta_{n-1} \in \mathbb{C}$, such that $\Psi(x) = x^n + \beta_{n-1}x^{n-1} + \cdots + \beta_1 x + \beta_0$. Then it holds that

$$e_1^T \Psi(T^{-1}AT) = [\beta_0 - \alpha_0, ..., \beta_{n-1} - \alpha_{n-1}].$$

(d) Let $F := e_n^T K(A, B)^{-1} \Psi(A)$. Then F solves the pole placement problem for (A, B) and $\mathcal{L}$, i.e., $\sigma(A - BF) = \mathcal{L}$.

Exercise 5.2: (Eigenvalues and pole placement)

Let $(A, B) \in \mathbb{R}^{n,n} \times \mathbb{R}^{n,1}$ be controllable and be given in Regelungsnormalform. Furthermore, let $\mathcal{L} = \{\mu_1, \dots, \mu_n\}$ be closed under complex conjugation, and $F = [f_1, \dots, f_n] \in \mathbb{R}^{1,n}$ such that $\sigma(A - BF) = \mathcal{L}$. Show:

(a) If $\lambda \in \mathcal{L} \cap \sigma(A)$, then every eigenvector of A for the eigenvalue λ is also eigenvector of $A - BF$ for the eigenvalue λ.

(b) $v = [v_1, \dots, v_n]^T \in \mathbb{C}^n$ is eigenvector of A if and only if

$$v_k = \lambda^{k-1} v_1 \text{ for } k = 1, \dots, n \text{ and } \sum_{k=1}^{n} (-\alpha_{k-1}) v_k = \lambda v_n.$$

Find an analogous formula for the eigenvectors of $A - BF$.

(c) Let $\lambda \in \mathcal{L} \setminus \sigma(A)$ and $x := (A - \lambda I)^{-1} B$. Show that: $Fx = 1$.
(Hint: show that xF has the eigenvalue 1, if $A - BF$ has the eigenvalue λ.)

(d) If $\lambda \in \mathcal{L} \setminus \sigma(A)$, then $(A - \lambda I)^{-1} B$ is an eigenvector of $A - BF$ for the eigenvalue λ.