Example 1.2.4 (Explicit Runge-Kutta Methods)

1. Explicit Euler method: $s = 1$, $p = 1$,

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array} \qquad \begin{array}{rcl} k_1 &=& f(t_m, u_m) \\ u_{m+1} &=& u_m + hk_1 \end{array}$$

2. Collatz method: $s = 2$, $p = 2$,

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{rcl} k_1 &=& f(t_m, u_m) \\ k_2 &=& f(t_m + \frac{h}{2}, u_m + \frac{h}{2}k_1) \\ u_{m+1} &=& u_m + hk_2 \end{array}$$

3. Heun method: $s = 2$, $p = 2$,

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{rcl} k_1 &=& f(t_m, u_m) \\ k_2 &=& f(t_m + h, u_m + hk_1) \\ u_{m+1} &=& u_m + \frac{h}{2}(k_1 + k_2) \end{array}$$

4. Heun method: $s = 3$, $p = 3$,

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array} \qquad \begin{array}{rcl} k_1 &=& f(t_m, u_m) \\ k_2 &=& f(t_m + \frac{1}{3}h, u_m + h\frac{1}{3}k_1) \\ k_3 &=& f(t_m + \frac{2}{3}h, u_m + h\frac{2}{3}k_2) \\ u_{m+1} &=& u_m + h[\frac{1}{4}k_1 + \frac{3}{4}k_3] \end{array}$$

5. Classical Runge-Kutta method: $s = 4$, $p = 4$,

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

6. 3/8-Rule: $s = 4$, $p = 4$,

$$\begin{array}{c|cccc} 0 & & & & \\ 1/3 & 1/3 & & & \\ 2/3 & -1/3 & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & 1/8 & 3/8 & 3/8 & 1/8 \end{array}$$

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