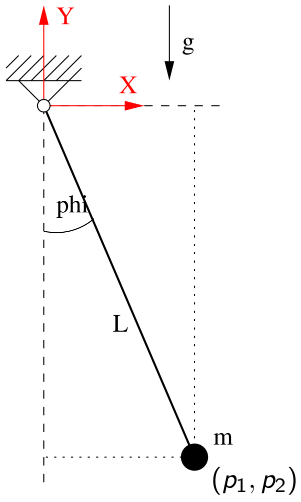
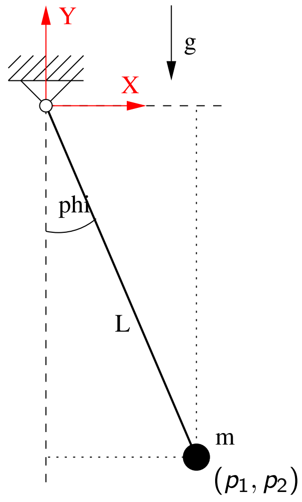
Simple pendulum



Introductory Example

Simple pendulum

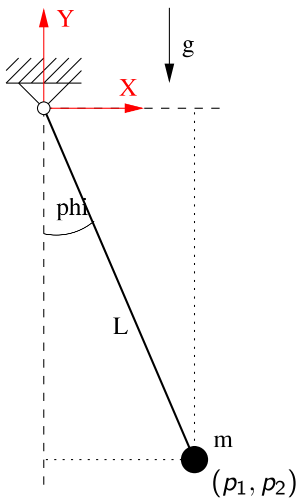


Equations of motion

$$\begin{aligned}
\dot{p}_1 &= v_1 \\
\dot{p}_2 &= v_2 \\
m\dot{v}_1 &= \qquad\quad -2p_1\lambda \\
m\dot{v}_2 &= -mg - 2p_2\lambda \\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}$$

Introductory Example

Simple pendulum



Equations of motion

$$\begin{aligned}
\dot{p}_1 &= v_1 \\
\dot{p}_2 &= v_2 \\
m\dot{v}_1 &= \qquad\quad -2p_1\lambda \\
m\dot{v}_2 &= -mg - 2p_2\lambda \\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}$$

with the hidden constraints ...

... of level 1

$$0 = 2p_1v_1 + 2p_2v_2$$

... of level 2

$$0 = 2v_1^2 + 2v_2^2 - 2v_2g - \frac{4}{m}(p_1^2 + p_2^2)\lambda$$

Introductory Example

Quasi-Linear DAE

$$E(x,t)\dot{x} = k(x,t)$$

with hidden constraints ...

... of level 1

$$0 = \widetilde{k}_c^1(x,t)$$

$$\vdots$$

... of level ν_c (maximal constraint level)

$$0 = \widetilde{k}_c^{\nu_c}(x,t)$$

Equations of motion

$$\begin{aligned}
\dot{p}_1 &= v_1 \\
\dot{p}_2 &= v_2 \\
m\dot{v}_1 &= \qquad\quad -2p_1\lambda \\
m\dot{v}_2 &= -mg - 2p_2\lambda \\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}$$

with the hidden constraints ...

... of level 1

$$0 = 2p_1v_1 + 2p_2v_2$$

... of level 2

$$0 = 2v_1^2 + 2v_2^2 - 2v_2g - \frac{4}{m}(p_1^2 + p_2^2)\lambda$$

Quasi-Linear DAE

$$E(x,t)\dot{x} = k(x,t)$$

with hidden constraints ...

... of level 1

$$0 = \widetilde{k}_c^1(x,t)$$

$$\vdots$$

... of level ν_c (maximal constraint level)

$$0 = \widetilde{k}_c^{\nu_c}(x,t)$$

Equations of motion

$$\begin{aligned} \dot{p}_1 &= v_1 \\ \dot{p}_2 &= v_2 \\ m\dot{v}_1 &= \quad\;\; -2p_1\lambda \\ m\dot{v}_2 &= -mg - 2p_2\lambda \\ 0 &= p_1^2 + p_2^2 - L^2 \end{aligned}$$

with the hidden constraints ...

... of level 1

$$0 = 2p_1v_1 + 2p_2v_2$$

... of level 2

$$0 = 2v_1^2 + 2v_2^2 - 2v_2 g - \frac{4}{m}(p_1^2 + p_2^2)\lambda$$

maximal constraint level 2

Quasi-Linear DAE

$$E(x,t)\dot{x} = k(x,t)$$

with initial values

$$x(0) = x_0$$

Quasi-Linear DAE

$$E(x,t)\dot{x} = k(x,t)$$

with initial values

$$x(0) = x_0$$

Discretization

e.g., with implicit Euler method

$$0 = E(x_i,t_i)(x_i - x_{i-1}) - hk(x_i,t_i)$$
$$(0 = F(x_i))$$

nonlinear equation for x_i,

Quasi-Linear DAE

$$E(x,t)\dot{x} = k(x,t)$$

with initial values

$$x(0) = x_0$$

Discretization

e.g., with implicit Euler method

$$0 = E(x_i,t_i)(x_i - x_{i-1}) - hk(x_i,t_i)$$
$$(0 = F(x_i))$$

nonlinear equation for x_i,

e.g., with Newtons method

$$\begin{aligned} J(x_i^k)\Delta^k &= -F(x_i^k) \\ x_i^{k+1} &= x_i^k + \Delta^k \end{aligned}$$

Simple Example

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= - 2p_1\lambda\\
m\dot v_2 &= -mg - 2p_2\lambda\\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}
$$

Simple Example

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= - 2p_1\lambda\\
m\dot v_2 &= -mg - 2p_2\lambda\\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}
$$

with initial values

$$
\begin{aligned}
p_1(0) &= L & p_2(0) &= 0\\
v_1(0) &= 0 & v_2(0) &= 0\\
\lambda(0) &= 0
\end{aligned}
$$

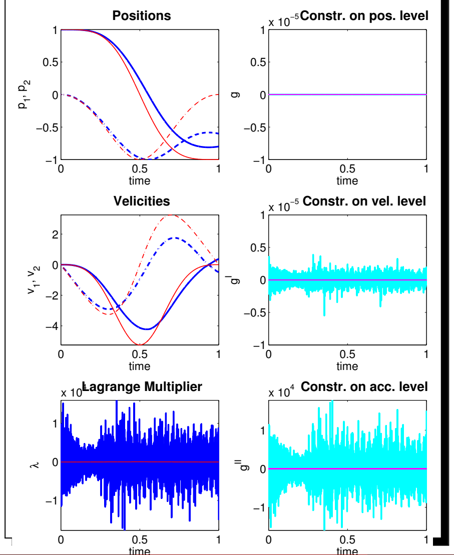
Simple Example

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= - 2p_1\lambda\\
m\dot v_2 &= -mg - 2p_2\lambda\\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}
$$

with initial values

$$
\begin{aligned}
p_1(0) &= L & p_2(0) &= 0\\
v_1(0) &= 0 & v_2(0) &= 0\\
\lambda(0) &= 0
\end{aligned}
$$

Numerical Results



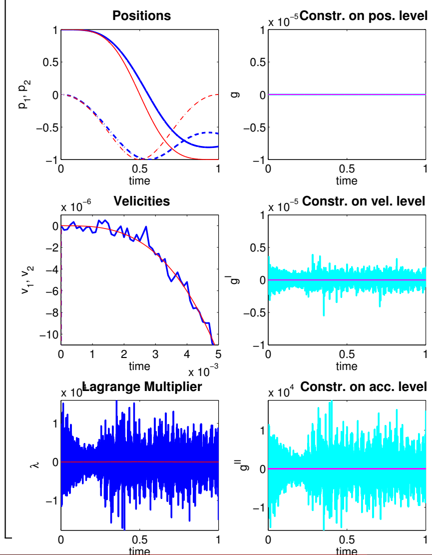
Simple Example

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= - 2p_1\lambda\\
m\dot v_2 &= -mg - 2p_2\lambda\\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}
$$

with initial values

$$
\begin{aligned}
p_1(0) &= L & p_2(0) &= 0\\
v_1(0) &= 0 & v_2(0) &= 0\\
\lambda(0) &= 0
\end{aligned}
$$

Numerical Results



Why we become in trouble with the numerical integration?

Why we become in trouble with the numerical integration?

Because of the constraints.

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

$\Rightarrow$ instabilities, oszillations
$\Rightarrow$ convergence problems,
$\Rightarrow$ order reduction of numerical algorithms,
$\Rightarrow$ inconsistencies.

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

$\Rightarrow$ instabilities, oszillations
$\Rightarrow$ convergence problems,
$\Rightarrow$ order reduction of numerical algorithms,
$\Rightarrow$ inconsistencies

A numerical integration of DAEs consisting hidden constraints, in general, is not to recommend.

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

⇒ instabilities, oszillations
⇒ convergence problems,
⇒ order reduction of numerical algorithms,
⇒ inconsistencies

We have to do ...

something

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

⇒ instabilities, oszillations
⇒ convergence problems,
⇒ order reduction of numerical algorithms,
⇒ inconsistencies

We have to do ...

Regularization

d-Index = Differentiation Index

Classical Approach - Index Reduction via the d-Index Concept

Basic Idea of the d-Index Concept

Replace the constraints by its derivatives and substitute differentiated unknowns as far as possible.

Simple Pendulum

equations of motion (d-index 3)

$$\begin{aligned}
\dot{p}_1 &= v_1 \\
\dot{p}_2 &= v_2 \\
m\dot{v}_1 &= - 2p_1\lambda \\
m\dot{v}_2 &= -mg - 2p_2\lambda \\
0 &= p_1^2 + p_2^2 - L^2
\end{aligned}$$

with the hidden constraints ...

... of level 1

$$0 = 2p_1v_1 + 2p_2v_2$$

... of level 2

$$0 = 2v_1^2 + 2v_2^2 - 2v_2g - \frac{4}{m}(p_1^2 + p_2^2)\lambda$$

Simple Pendulum

d-index 2 formulation

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= \qquad\; -2p_1\lambda\\
m\dot v_2 &= -mg-2p_2\lambda\\
0 &= 2p_1v_1+2p_2v_2
\end{aligned}
$$

with the hidden constraints ...
... of level 1

$$0=2v_1^2+2v_2^2-2v_2g-\frac{4}{m}(p_1^2+p_2^2)\lambda$$

Simple Pendulum

d-index 2 formulation

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= \qquad\; -2p_1\lambda\\
m\dot v_2 &= -mg-2p_2\lambda\\
0 &= 2p_1v_1+2p_2v_2
\end{aligned}
$$

with the hidden constraints ...
... of level 1

$$0=2v_1^2+2v_2^2-2v_2g-\frac{4}{m}(p_1^2+p_2^2)\lambda$$

and removed constraints

$$0=p_1^2+p_2^2-L^2$$

Simple Pendulum

d-index 2 formulation

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= \qquad\; -2p_1\lambda\\
m\dot v_2 &= -mg-2p_2\lambda\\
0 &= 2p_1v_1+2p_2v_2
\end{aligned}
$$

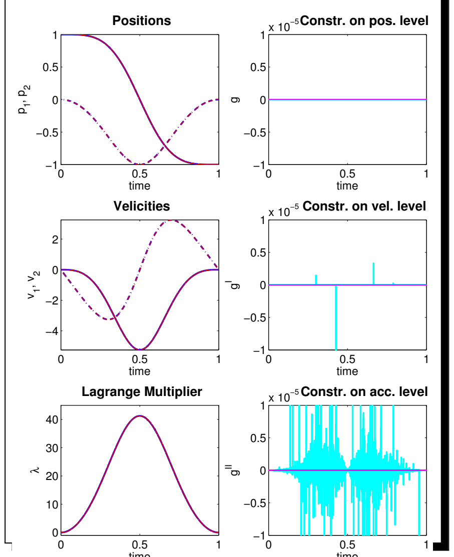
with the hidden constraints ...
... of level 1

$$0=2v_1^2+2v_2^2-2v_2g-\frac{4}{m}(p_1^2+p_2^2)\lambda$$

and removed constraints

$$0=p_1^2+p_2^2-L^2$$

Numerical Results



Simple Pendulum

d-index 2 formulation

$$
\begin{aligned}
\dot p_1 &= v_1\\
\dot p_2 &= v_2\\
m\dot v_1 &= \qquad\; -2p_1\lambda\\
m\dot v_2 &= -mg-2p_2\lambda\\
0 &= 2p_1v_1+2p_2v_2
\end{aligned}
$$

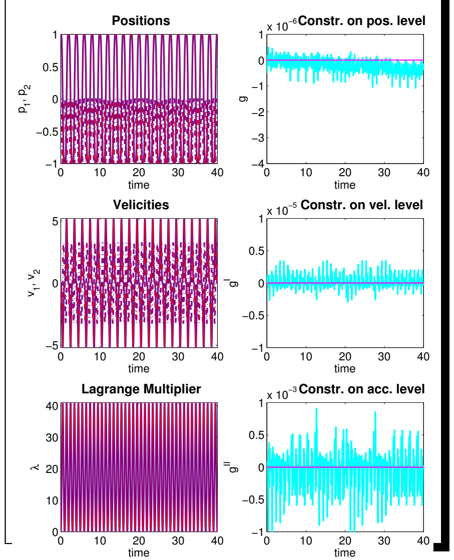
with the hidden constraints ...
... of level 1

$$0=2v_1^2+2v_2^2-2v_2g-\frac{4}{m}(p_1^2+p_2^2)\lambda$$

and removed constraints

$$0=p_1^2+p_2^2-L^2$$

Numerical Results



Simple Pendulum

d-index 2 formulation

$$\begin{aligned} \dot p_1 &= v_1 \\ \dot p_2 &= v_2 \\ m\dot v_1 &= - 2p_1\lambda \\ m\dot v_2 &= -mg - 2p_2\lambda \\ 0 &= 2p_1v_1 + 2p_2v_2 \end{aligned}$$

with the hidden constraints ...
... of level 1

$$0 = 2v_1^2 + 2v_2^2 - 2v_2 g - \frac{4}{m}(p_1^2 + p_2^2)\lambda$$

and removed constraints

$$0 = p_1^2 + p_2^2 - L^2$$

Simple Pendulum

d-index 1 formulation

$$\begin{aligned} \dot p_1 &= v_1 \\ \dot p_2 &= v_2 \\ m\dot v_1 &= - 2p_1\lambda \\ m\dot v_2 &= -mg - 2p_2\lambda \\ 0 &= 2v_1^2 + 2v_2^2 - 2v_2 g \\ &\quad - \frac{4}{m}(p_1^2 + p_2^2)\lambda \end{aligned}$$

with no hidden constraints

Simple Pendulum

d-index 1 formulation

$$\begin{aligned} \dot p_1 &= v_1 \\ \dot p_2 &= v_2 \\ m\dot v_1 &= - 2p_1\lambda \\ m\dot v_2 &= -mg - 2p_2\lambda \\ 0 &= 2v_1^2 + 2v_2^2 - 2v_2 g \\ &\quad - \frac{4}{m}(p_1^2 + p_2^2)\lambda \end{aligned}$$

with no hidden constraints
but removed constraints

$$0 = p_1^2 + p_2^2 - L^2$$
$$0 = 2p_1v_1 + 2p_2v_2$$

Simple Pendulum

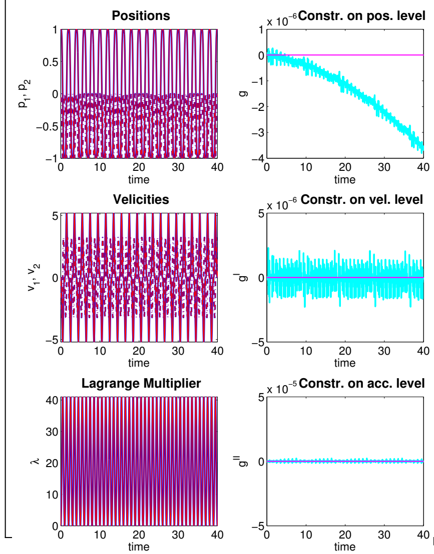
d-index 1 formulation

$$\begin{aligned} \dot p_1 &= v_1 \\ \dot p_2 &= v_2 \\ m\dot v_1 &= - 2p_1\lambda \\ m\dot v_2 &= -mg - 2p_2\lambda \\ 0 &= 2v_1^2 + 2v_2^2 - 2v_2 g \\ &\quad - \frac{4}{m}(p_1^2 + p_2^2)\lambda \end{aligned}$$

with no hidden constraints
but removed constraints

$$0 = p_1^2 + p_2^2 - L^2$$
$$0 = 2p_1v_1 + 2p_2v_2$$

Numerical Results



Index Reduction via the d-Index Concept – Example

Simple Pendulum

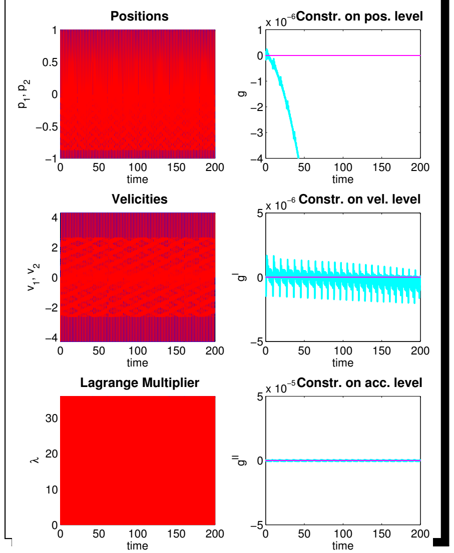
d-index 1 formulation

$$\begin{aligned} \dot{p}_1 &= v_1 \\ \dot{p}_2 &= v_2 \\ m\dot{v}_1 &= \quad -2p_1\lambda \\ m\dot{v}_2 &= -mg - 2p_2\lambda \\ 0 &= 2v_1^2 + 2v_2^2 - 2v_2 g \\ & \qquad -\frac{4}{m}(p_1^2 + p_2^2)\lambda \end{aligned}$$

with no hidden constraints
but removed constraints

$$0 = p_1^2 + p_2^2 - L^2$$
$$0 = 2p_1 v_1 + 2p_2 v_2$$

Numerical Results



Index Reduction via the d-Index Concept – Summary

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

⇒ instabilities, oszillations
⇒ convergence problems,
⇒ order reduction of numerical algorithms,
⇒ inconsistencies.

Index Reduction via the d-Index Concept – Summary

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

⇒ instabilities, oszillations
⇒ convergence problems,
⇒ order reduction of numerical algorithms,
⇒ inconsistencies.

removed constraints

⇒ drift, since the solution is no longer restricted into the set of consistency

Index Reduction via the d-Index Concept – Summary

Why we become in trouble with the numerical integration?

Because of the constraints.

hidden constraints

⇒ instabilities, oszillations
⇒ convergence problems,
⇒ order reduction of numerical algorithms,
⇒ inconsistencies.

removed constraints

⇒ drift, since the solution is no longer restricted into the set of consistency

Up to now we dont have a regularization technique.
We only have an index reduction technique.