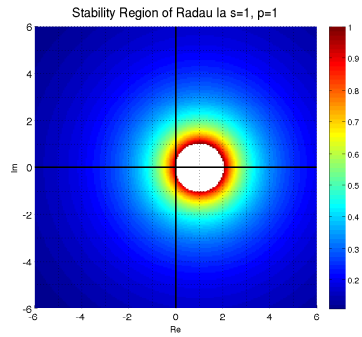
Example 1.2.17 (Implicit Runge-Kutta Methods)

1. Radau Ia: $s = 1$, $p = 1$,
   (Class: Radau Ia)

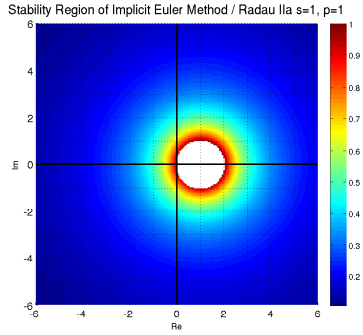
$$\begin{array}{c|c} 0 & 1 \\ \hline & 1 \end{array} \qquad \begin{aligned} k_1 &= f(t_m, u_m + hk_1) \\ u_{m+1} &= u_m + hk_1 \end{aligned}$$



2. Implicit Euler method: $s = 1$, $p = 1$,
   (Class: Radau IIa)

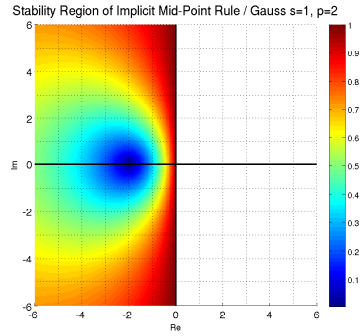
$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \begin{aligned} k_1 &= f(t_m + h, u_m + hk_1) \\ u_{m+1} &= u_m + hk_1 \end{aligned}$$

Stability function: $R(z) = \frac{1}{1-z}$



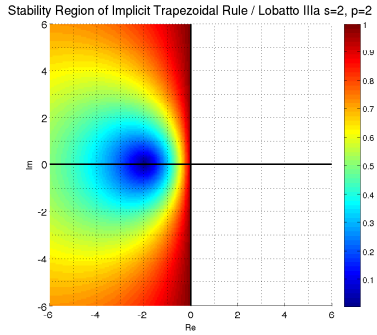
3. Implicit mid-point rule: $s = 1$, $p = 2$,
   (Class: Gauss)

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array} \qquad \begin{aligned} k_1 &= f(t_m + \tfrac{1}{2}h, u_m + h\tfrac{1}{2}k_1) \\ u_{m+1} &= u_m + hk_1 \end{aligned}$$



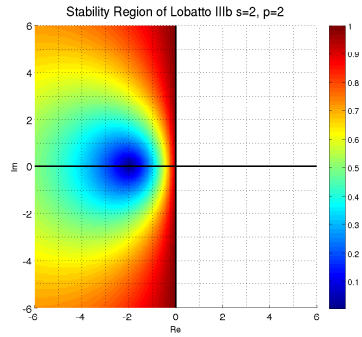
4. Trapezoidal Rule: $s = 2$, $p = 2$,
   (Class: Lobatto IIIa)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{aligned} k_1 &= f(t_m, u_m) \\ k_2 &= f(t_m + h, u_m + h(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2)) \\ u_{m+1} &= u_m + h(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2) \end{aligned}$$



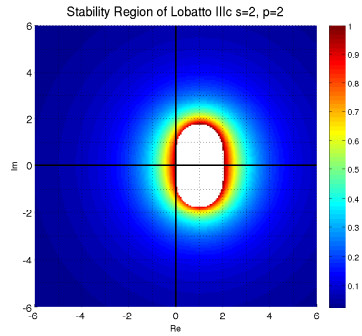
5. Lobatto IIIb: $s = 2$, $p = 2$,
   (Class: Lobatto IIIb)

$$\begin{array}{c|cc} 0 & 1/2 & 0 \\ 1 & 1/2 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{aligned} k_1 &= f(t_m, u_m + h\tfrac{1}{2}k_1) \\ k_2 &= f(t_m + h, u_m + h\tfrac{1}{2}k_1) \\ u_{m+1} &= u_m + h(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2) \end{aligned}$$



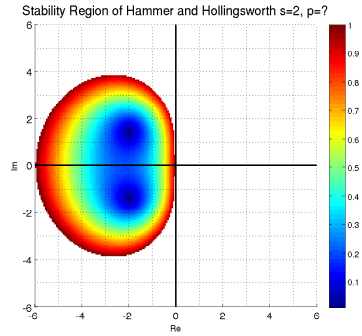
6. Lobatto IIIc: $s = 2$, $p = 2$,
(Class: Lobatto IIIc)

$$\begin{array}{c|cc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{aligned} k_1 &= f(t_m, u_m + h[\tfrac{1}{2}k_1 - \tfrac{1}{2}k_2]) \\ k_2 &= f(t_m + h, u_m + h[\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2]) \\ u_{m+1} &= u_m + h[\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2] \end{aligned}$$



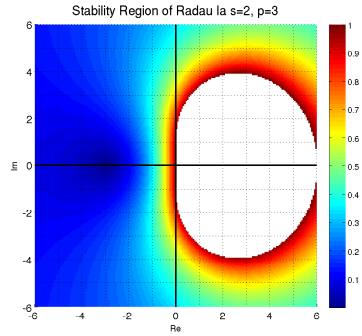
7. Hammer & Hollingsworth: $s = 2$, $p =$ [Excercise],

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 1/3 & 1/3 \\ \hline & 1/4 & 3/4 \end{array} \qquad \begin{aligned} k_1 &= f(t_m, u_m) \\ k_2 &= f(t_m + \tfrac{2}{3}h, u_m + h(\tfrac{1}{3}k_1 + \tfrac{1}{3}k_2)) \\ u_{m+1} &= u_m + h(\tfrac{1}{4}k_1 + \tfrac{3}{4}k_2) \end{aligned}$$



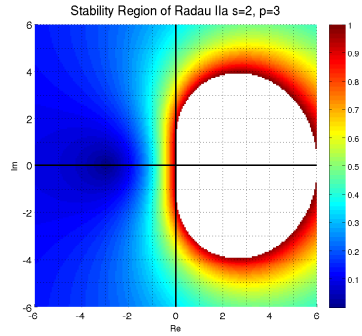
8. Radau Ia: $s = 2$, $p = 3$,
(Class: Radau Ia)

$$\begin{array}{c|cc} 0 & 1/4 & -1/4 \\ 2/3 & 1/4 & 5/12 \\ \hline & 1/4 & 3/4 \end{array} \qquad \begin{aligned} k_1 &= f(t_m, u_m + h(\tfrac{1}{4}k_1 - \tfrac{1}{4}k_2)) \\ k_2 &= f(t_m + \tfrac{2}{3}h, u_m + h(\tfrac{1}{4}k_1 + \tfrac{5}{12}k_2)) \\ u_{m+1} &= u_m + h(\tfrac{1}{4}k_1 + \tfrac{3}{4}k_2) \end{aligned}$$



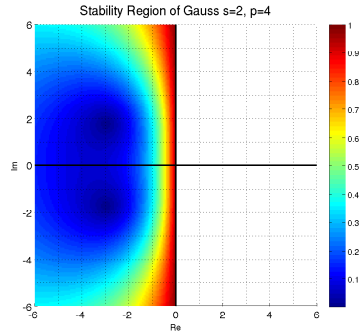
9. Ehle Method: $s = 2$, $p = 3$,
(Class: Radau IIa)

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array} \qquad \begin{aligned} k_1 &= f(t_m + \tfrac{1}{3}h, u_m + h(\tfrac{5}{12}k_1 - \tfrac{1}{12}k_2)) \\ k_2 &= f(t_m + h, u_m + h(\tfrac{3}{4}k_1 + \tfrac{1}{4}k_2)) \\ u_{m+1} &= u_m + h(\tfrac{3}{4}k_1 + \tfrac{1}{4}k_2) \end{aligned}$$



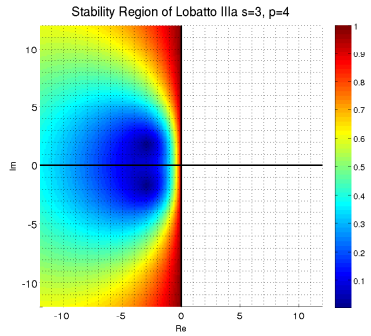
10. Hammer & Hollingsworth: $s = 2$, $p = 4$,
(Class: Gauss)

$$\begin{array}{c|cc} (3-\sqrt{3})/6 & 1/4 & (3-2\sqrt{3})/12 \\ (3+\sqrt{3})/6 & (3+2\sqrt{3})/12 & 1/4 \\ \hline & 1/2 & 1/2 \end{array}$$



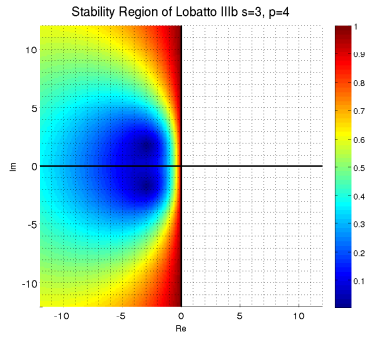
11. Ehle Method: $s = 3$, $p = 4$,
(Class: Lobatto IIIa)

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 5/24 & 1/3 & -1/24 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$



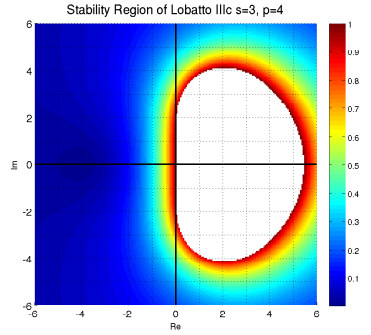
12. Lobatto IIIb: $s = 3$, $p = 4$,
(Class: Lobatto IIIb)

$$\begin{array}{c|ccc} 0 & 1/6 & -1/6 & 0 \\ 1/2 & 1/6 & 1/3 & 0 \\ 1 & 1/6 & 5/6 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$



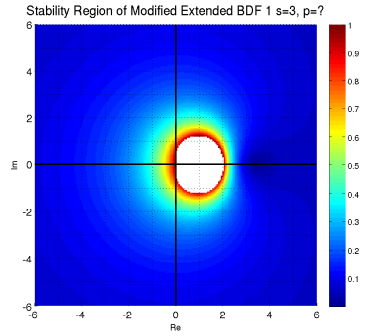
13. Lobatto IIIc: $s = 3$, $p = 4$,
(Class: Lobatto IIIc)

$$\begin{array}{c|ccc} 0 & 1/6 & -1/3 & 1/6 \\ 1/2 & 1/6 & 5/12 & -1/12 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$



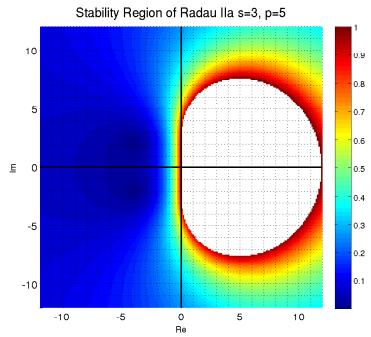
14. Modified Extended BDF 1: $s = 3$, $p =$[Exercise],
(Class: SDIRK)

$$\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1/2 & -1/2 & 1 \\ \hline & 1/2 & -1/2 & 1 \end{array}$$



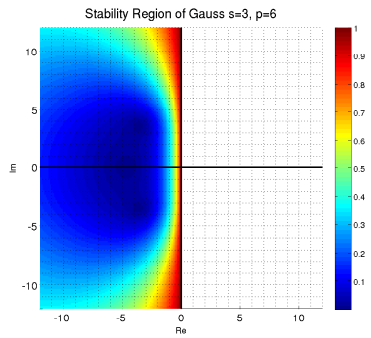
15. Radau IIa: $s = 3$, $p = 5$,
(Class: Radau IIa)

$$\begin{array}{c|ccc} \frac{4-\sqrt{6}}{10} & \frac{88-7\sqrt{6}}{360} & \frac{296-169\sqrt{6}}{1800} & \frac{-2+3\sqrt{6}}{225} \\ \frac{4+\sqrt{6}}{10} & \frac{296+169\sqrt{6}}{1800} & \frac{88+7\sqrt{6}}{360} & \frac{-2-3\sqrt{6}}{225} \\ 1 & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \\ \hline & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \end{array}$$



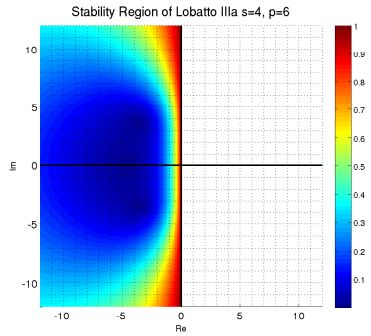
16. Gauss: $s = 3$, $p = 6$,
(Class: Gauss)

$$\begin{array}{c|ccc} 1/2-\sqrt{15}/10 & 5/36 & 2/9-\sqrt{15}/15 & 5/36-\sqrt{15}/30 \\ 1/2 & 5/36+\sqrt{15}/24 & 2/9 & 5/36-\sqrt{15}/24 \\ 1/2+\sqrt{15}/10 & 5/36+\sqrt{15}/30 & 2/9+\sqrt{15}/15 & 5/36 \\ \hline & 5/18 & 4/9 & 5/18 \end{array}$$



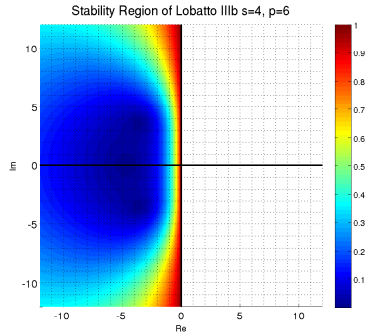
17. Lobatto IIIa: $s = 4$, $p = 6$,
(Class: Lobatto IIIa)

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{5-\sqrt{5}}{10} & \frac{11+\sqrt{5}}{120} & \frac{25-\sqrt{5}}{120} & \frac{25-13\sqrt{5}}{120} & \frac{-1+\sqrt{5}}{120} \\
\frac{5+\sqrt{5}}{10} & \frac{11-\sqrt{5}}{120} & \frac{25+13\sqrt{5}}{120} & \frac{25+\sqrt{5}}{120} & \frac{-1-\sqrt{5}}{120} \\
1 & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \\
\hline
 & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}
\end{array}$$



18. Lobatto IIIb: $s = 4$, $p = 6$,
(Class: Lobatto IIIb)

$$\begin{array}{c|cccc}
0 & \frac{1}{12} & \frac{-1-\sqrt{5}}{24} & \frac{-1+\sqrt{5}}{24} & 0 \\
\frac{5-\sqrt{5}}{10} & \frac{1}{12} & \frac{25+\sqrt{5}}{120} & \frac{25-13\sqrt{5}}{120} & 0 \\
\frac{5+\sqrt{5}}{10} & \frac{1}{12} & \frac{25+13\sqrt{5}}{120} & \frac{25-\sqrt{5}}{120} & 0 \\
1 & \frac{1}{12} & \frac{11-\sqrt{5}}{24} & \frac{11+\sqrt{5}}{24} & 0 \\
\hline
 & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}
\end{array}$$



◁