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Exercise sheet 10

Deadline: Thursday, July 5th, 2007, 08:30 h in MA-313

Exercise 1: **4 points**

Suppose we want to solve the vehicle routing problem on graph $G = (V, E)$ with $V = J \cup \{d\}$, where d is the depot and the distances are given by the length function $l_e \colon E \to \mathbb{R}_{\geq 0}$. Consider the MIP formulation

$$
\begin{array}{llll}
\min & \sum_{e \in E} l_e y_e & & \\
s.t. & -y_e + \sum_{t \in \mathcal{T}'_k} \alpha^t_e x_t \leq 0, & \forall e \in E \\
& \sum_{t \in \mathcal{T}'_k} \alpha^t_j x_t = 1, & \forall j \in J \\
& y(\delta(j)) = 2, & \forall j \in J \\
& y_e \in \{0,1,2\}, & \forall e \in E \\
& x_t \in [0,1], & \forall t \in \mathcal{T}'_k
\end{array}
$$

where $\mathcal{T}'_k$ is the set of tours visiting at most k customers with repetitions of customers allowed and α^t_e (α^t_j) counts how often edge e (node j) is traversed in $t \in \mathcal{T}'_k$.

a) Show that if all y_e are integer, then there is a set of tours from $\mathcal{T}'_k$ that is compatible with the values of y_e.

b) Does y_e integer for all e imply that the x_t are integer, too?

Exercise 2: **4 points**

Consider the vehicle routing instance given by the complete graph on $\{1, 2, 3, 4\}$ (we assume $d = 1$), the symmetric length function specified by the matrix

$$
\begin{pmatrix}
0 & 1 & 1 & 1 \\
 & 0 & 2 & 1 \\
 & & 0 & 1 \\
 & & & 0
\end{pmatrix}
$$

and hop limit $k = 3$. Suppose we want to solve this instance using the formulation from Exercise 1 using column generation and that in one pricing iteration we get the dual prices

$$
\pi_{10} = -5/6, \pi_{20} = -2/3, \pi_{21} = -4/3, \pi_{30} = -1/3, \pi_{31} = 0, \pi_{32} = 0,
$$
$$
\sigma_1 = 5/3, \sigma_2 = 2/3, \sigma_3 = 0,
$$

where π_e are the dual variables for the first block of constraints and σ_j for the second block of constraints. What is the tour with the most negative reduced cost?

Exercise 3: **4 points**

For a graph $G = (V, E)$ the matroid $(E, \mathcal{I})$ with $\mathcal{I}$ defined by

$$\mathcal{I} := \{F \subseteq E \mid F \text{ is a forest}\}$$

is called the *graphical matroid* corresponding to G. Show that $F \subseteq E$ is inseparable if and only if $(V(F), F)$ is 2-node-connected.

Note: In any matroid $(E, \mathcal{I})$ a subset $F \subseteq E$ is called *separable* if there are $\emptyset \neq F', F'' \subseteq F$ with $F' \cap F'' = \emptyset$, $F' \cup F'' = F$ and

$$r(F) = r(F') + r(F'').$$

Otherwise, F is called *inseparable.*

Exercise 4: **4 points**

The rank function of a matroid is always supermodular. For a finite set E, a function $f \colon 2^E \to \mathbb{R}$ is called *supermodular* if it satisfies

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$$

for all subsets $X, Y \subseteq 2^E$. Show that this is equivalent to requiring

$$f(X \cup \{x, y\}) - f(X \cup \{y\}) \leq f(X \cup \{x\}) - f(X)$$

for all $X \subseteq 2^E$ and $x, y \in E$.