Technische Universität Berlin
Institut für Mathematik

ADM III – Advanced Methods for Integer Linear Programming
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# Exercise sheet 6

Deadline: Thursday, June 7th, 2007, 08:30 h in MA-313

**Exercise 1:** **4 points**

Consider the integer program

$$\begin{aligned} \min\ & x_{n+1} \\ 2x_1 + 2x_2 + \ldots + 2x_n + x_{n+1} &= n \\ x &\in \{0,1\}^{n+1} \ . \end{aligned}$$

Show that any branch-and-bound algorithm that, at each branch-and-bound node,

- only uses the linear programming relaxation to compute a lower bound (without adding further valid inequalities) and
- branches on a variable with fractional LP solution value

requires the enumeration of an exponential (with respect to $n$) number of branch-and-bound nodes when $n$ is odd.

**Exercise 2:** **4 points**

Show that the following two matrices are totally unimodular.

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

**Exercise 3:** **4 points**

The polytope $\text{MATCH}^*(G) \subseteq \mathbb{R}^E$ of all perfect matchings in a graph $G = (V, E)$ with $|V|$ even can be completely described by linear equalities and inequalities as

$$\begin{aligned} \text{MATCH}^*(G) := \{x \in \mathbb{R}^E : (1) \quad & \sum_{e \in \delta(v)} x_e = 1 && \forall\, v \in V \\ (2) \quad & \sum_{e \in \delta(W)} x_e \geq 1 && \forall\, W \subset V : \ |W| \text{ odd}, |W| \geq 3 \\ (3) \quad & x_e \geq 0 && \forall\, e \in E\} \ . \end{aligned}$$

(i) Show that the system (1)–(3) is not TDI for all graphs with $|V|$ even.

(ii) Given an alternative complete description of $\mathrm{MATCH}^*(G)$ with linear equalities and inequalities that is TDI, and prove that this description is TDI.

**Exercise 4:** **4 points**

Prove the following theorem using polyhedral methods:

**Theorem [Lucchesi and Younger, 1978]** *The maximum number of edge-disjoint directed cuts in a digraph equals the minimum cardinality of an edge set that contains at least one edge of each directed cut.*

(A directed cut in the digraph $G = (V, E)$ is an arc set $\delta^+(W) := \{ij \in E \mid i \in W, j \in V \setminus W\}$ for some set $\emptyset \neq W \subsetneq V$.)

Hint: Consider the matrix $A$ whose columns are indexed by the edges and whose rows are the incidence vectors of all directed cuts. With this matrix, formulate the two problems of finding the maximum number of edge-disjoint directed cuts and of finding a minimum cardinality edge set that contains at least one edge of each directed cut as a pair of dual integer linear programs. Then show that these programs are TDI.