Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Discrete Geometry

(Kombinatorische Geometrie I)

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Exercise Sheet 1

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Exercise 1. **4 points**

Consider the polytope

$$P := \mathsf{conv}\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 2 \\ 2 \end{pmatrix} \right\} \subset \mathbb{R}^4.$$

Show that P is 3-dimensional and give an affine map $\Theta : \mathbb{R}^4 \to \mathbb{R}^3$ that induces an affine equivalence.
Sketch $\Theta(P)$. How many vertices, edges and facets does P have?

Exercise 2. **4 points**

(a) Show that for every d-dimensional polytope $P \subset \mathbb{R}^e$ there exists a polytope $P' \subset \mathbb{R}^d$ such that P and P' are affinely equivalent.

(b) Show that affine equivalence of polytopes is indeed an equivalence relation.

Exercise 3. **4 points**

(a) Show that every simple and simplicial polytope of dimension $d \geq 3$ is necessarily a d-simplex.

(b) Show that if a polytope is k-neighbourly then every $(2k-1)$-face is a simplex. Conclude that if a d-polytope is $(\lfloor d/2 \rfloor + 1)$-neighbourly then it is a simplex.

Exercise 4. **4 points**

Show that the centrally-symmetric polytope

$$\mathsf{conv}\left\{ \mathbf{x} \in \{-1, +1\}^{2k-1} \mid -1 \leq x_1 + \ldots + x_{2k-1} \leq 1 \right\}$$

is combinatorially equivalent to the $(2k-1)$-dimensional hypersimplex $\Delta_{2k-1}(k)$. Which polytope do we get for $k = 2$? Count the vertices and facets of the above polytope for $k = 3$.

please turn over

Exercise 5. **(Tutorial)**

The points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$ are *affinely independent* if their affine span

$$\left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \;\middle|\; \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

is an affine subspace of dimension $k-1$.

(a) Show that $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$ are affinely independent if and only if the points $\mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{x}_k - \mathbf{x}_1$ are linearly independent.

(b) Show that $\mathbf{x}_1, \dots, \mathbf{x}_k$ are affinely independent if and only if the points

$$\begin{pmatrix} 1 \\ \mathbf{x}_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ \mathbf{x}_k \end{pmatrix} \in \mathbb{R}^{d+1}$$

are linearly independent, that is, the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \mathbf{x}_1 & \cdots & \mathbf{x}_k \end{pmatrix}$$

has rank k.

(c) Sketch the situation in (b) for three affinely independent points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^2$.

(d) Show that an affine map $\mathbb{R}^d \to \mathbb{R}^e$ is uniquely determined by the images of $d+1$ affinely independent points.

(e) Give examples of 3-dimensional polytopes that are combinatorially equivalent but not affinely.