Discrete Geometry

(Kombinatorische Geometrie I)

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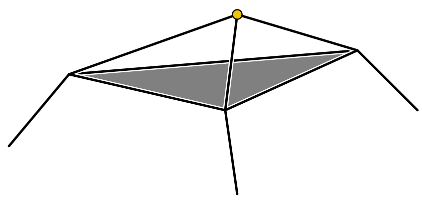
Exercise Sheet 2

Deadline: 5 May 2008

Exercise 6. **4 points**

Recall: A *d-dimensional stacked polytope on n vertices* arises from a d-simplex by carrying out $n-d-1$ stacking operations, where a *stacking operation* amounts to "gluing" a pyramid over a facet onto the facet in such a way that the result is again convex.



(a) Calculate the f-vector of a 4-dimensional stacked polytope with n vertices.

(b) Give two stacked polytopes (of your favourite dimension) with the same number of vertices which are *not* combinatorially equivalent.

Exercise 7. **4 points**

(a) Show that every polytope is the image of a standard simplex under some projection.

(b) Show that every centrally-symmetric polytope (with respect to the origin) is the image of a standard crosspolytope under some projection.

Exercise 8. **4 points**

Prove *Radon's theorem*: If V is a set of $d+2$ points in $\mathbb{R}^d$, there exist disjoint non-empty subsets $V_1, V_2 \subseteq V$ such that $\mathsf{conv}\, V_1 \cap \mathsf{conv}\, V_2 \neq \emptyset$.
Hint: Carathéodory's theorem.

Exercise 9. **4 points**

Sketch all 3-dimensional 0/1-polytopes up to affine equivalence. How many affine and how many combinatorial equivalence classes are there?

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Exercise 10. **(Tutorial)**

(a) The *cyclic moment curve* in $\mathbb{R}^{2k}$ is defined by

$$\mathbf{p}(t) = (\cos t,\, \sin t,\, \cos 2t,\, \sin 2t,\, \ldots,\, \cos kt,\, \sin kt)^\top.$$

For $n \geq 2k+1$ and $0 \leq t_1 < \ldots < t_n < 2\pi$ define

$$P := \mathsf{conv}\{\mathbf{p}(t_i) \mid 1 \leq i \leq n\}.$$

Show that P is combinatorially equivalent to a $(2k)$-dimensional cyclic polytope with n vertices.
Hint:

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \cos\nu_0 & \cos\nu_1 & & \cos\nu_{2k} \\ \sin\nu_0 & \sin\nu_1 & \cdots & \sin\nu_{2k} \\ \vdots & \vdots & & \vdots \\ \cos k\nu_0 & \cos k\nu_1 & & \cos k\nu_{2k} \\ \sin k\nu_0 & \sin k\nu_1 & \cdots & \sin k\nu_{2k} \end{pmatrix} = 4^{k^2} \prod_{0 \leq i < j \leq 2k} \sin\frac{\nu_j - \nu_i}{2}$$

(b) Show that the number of facets of a cyclic polytope with n vertices is

$$\begin{aligned} f_{d-1}(\mathcal{C}_d(n)) &= \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n-1-\lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor} \\ &= \begin{cases} \frac{n}{n-k}\binom{n-k}{k} & \text{for } d = 2k \text{ even} \\ 2\binom{n-k-1}{k} & \text{for } d = 2k+1 \text{ odd} \end{cases} \end{aligned}$$

Hint: Show that the number of ways in which $2k$ elements can be chosen from $\{1, \ldots, n\}$ in "even blocks of adjacent elements" is $\binom{n-k}{k}$.

(c) Is the polytope $(\Delta_2 \times \Delta_2)^\Delta$ combinatorially equivalent to $\mathcal{C}_4(6)$?
Hint: You can check if you're right with `polymake`: The client that constructs the dual (or polar) of a polytope is called `polarize`. (For this the polytope that is about to be polarized, has to be full-dimensional and centered, the last of which can be achieved by the `center` client.)