Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Discrete Geometry

(Kombinatorische Geometrie I)

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Exercise Sheet 3

Deadline: 12 May 2008

Exercise 11. **4 points**

Find an $\mathcal{H}$-description of the polytope

$$P := \mathsf{conv}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

by applying Fourier-Motzkin elimination.

Exercise 12. **4 points**

(a) Draw a Hasse diagram of the face lattice of a prism over a triangle. How many maximal chains does it contain? (Count smartly...!)

(b) Give a poset that satisfies all of the conditions of a polytope face lattice, but does not correspond to some polytope. Can you find some other geometric object that it corresponds to?

Exercise 13. **4 points**

State and prove a Farkas lemma for systems of the form $A\mathbf{x} \le \mathbf{z}$, $\mathbf{x} \ge \mathbf{0}$.

Exercise 14. **4 points**

For a polytope P and a face F of P we define the *face figure* $P/F := (F^\diamond)^\Delta$, that is the polar of the face of P^Δ that corresponds to F.
Show that P/F is a polytope of dimension $\mathsf{dim}(P/F) = \mathsf{dim}(P) - \mathsf{dim}(F) - 1$ and describe the face lattice $L(P/F)$ in terms of $L(P)$ and the face $F \in L(P)$.
Describe a direct geometric construction for P/F as an iterated vertex figure that generalises the definition of the vertex figure.

please turn over

Exercise 15. **(Tutorial)**

Recall: A *(polyhedral) cone* in $\mathbb{R}^d$ is the conical hull of a finite set of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^d$:

$$\mathsf{cone}(A) \;=\; \{t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n \mid t_i \geq 0\} \;=\; \{A\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\}$$

where A is the matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

(a) Give a geometric interpretation of the Farkas Lemma II:
If $A \in \mathbb{R}^{n\times m}$ and $\mathbf{z} \in \mathbb{R}^n$ then
either there is some $\mathbf{x} \in \mathbb{R}^m$ with $A\mathbf{x} = \mathbf{z}$, $\mathbf{x} \geq \mathbf{0}$,
or there is some $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^\top A \geq \mathbf{0}^\top$ and $\mathbf{c}^\top\mathbf{z} < 0$,
but not both.

(b) Let $P \subset \mathbb{R}^d$ be a convex set. The *recession cone* of P is the cone

$$\mathsf{rec}(P) \;:=\; \{\mathbf{y} \in \mathbb{R}^d \mid \mathbf{x} + t\mathbf{y} \in P \text{ for all } \mathbf{x} \in P, t \geq 0\}.$$

What is the recession cone of a polytope?
The *homogenisation* of P is defined by

$$\mathsf{homog}(P) \;:=\; \left\{t\begin{pmatrix}1\\ \mathbf{x}\end{pmatrix} \mid \mathbf{x} \in P, t > 0\right\} + \left\{\begin{pmatrix}0\\ \mathbf{y}\end{pmatrix} \mid \mathbf{y} \in \mathsf{rec}(P)\right\}.$$

Show that $\mathsf{homog}(P)$ is a polyhedral cone in $\mathbb{R}^{d+1}$ and

$$\begin{aligned}\mathsf{homog}(P) \;&=\; \mathsf{cone}\begin{pmatrix}\mathbf{1}^\top & \mathbf{0}^\top\\ V & Y\end{pmatrix}\\ &=\; \left\{\mathbf{x} \in \mathbb{R}^{d+1} \mid \begin{pmatrix}-1 & \mathbf{0}^\top\\ -\mathbf{z} & A\end{pmatrix} \leq \begin{pmatrix}0\\ \mathbf{0}\end{pmatrix}\right\}\end{aligned}$$

where

- V is the matrix whose columns are exactly the vertices of P,
- Y is the matrix containing the extreme rays of $\mathsf{rec}(P)$: $\mathsf{rec}(P) = \mathsf{cone}(Y)$,
- A and $\mathbf{z}$ define an $\mathcal{H}$-representation of P: $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{z}\}$.

(c) *Carathéodory's Theorem* (a version for cones and one for polytopes):

(a) Let $C \subset \mathbb{R}^d$ be a d-dimensional cone with extreme rays $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and $\mathbf{x} \in C$. Then $\mathbf{x}$ can be written as a conical combination of at most d extreme rays of C.

(b) Let $P \subset \mathbb{R}^d$ be a d-polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{x} \in P$. Then $\mathbf{x}$ can be written as a convex combination of at most $d+1$ vertices of P.