Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
Sommersemester 2008

Discrete Geometry

(Kombinatorische Geometrie I)

Prof. Günter M. Ziegler

Axel Werner

Exercise Sheet 4

Deadline: 19 May 2008

Exercise 16. **4 points**

(a) Consider the 2-polytope

$$P := \mathrm{conv}\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

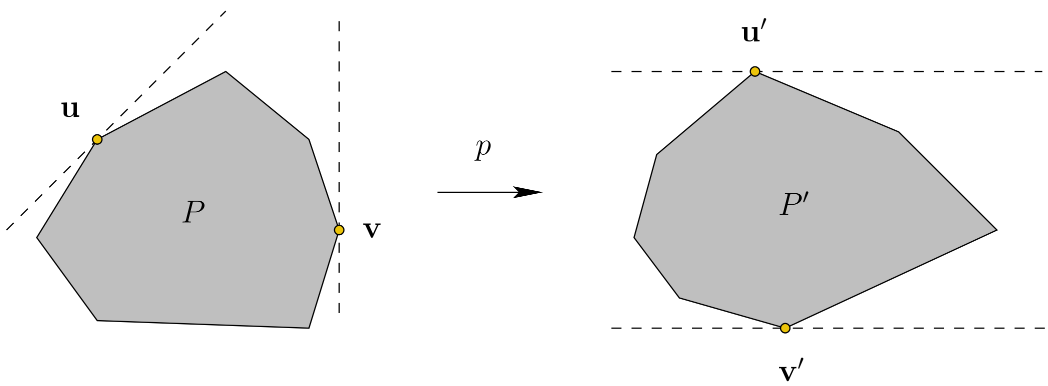
Calculate the vertices of P^Δ and sketch both P and P^Δ. Now translate to get the polytope $Q := P + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Again calculate the vertices of Q^Δ and sketch it.

old version: $Q := P - \binom{1}{1}$

(b) Prove: If P is a polytope with $\mathbf{0} \notin P$ then P^Δ is unbounded.

Exercise 17. **4 points**

Let $P \subset \mathbb{R}^d$ be a polytope and $\mathbf{u}, \mathbf{v}$ two distinct vertices of P. Show that there is a projective transformation p such that the vertices $\mathbf{u}' := p(\mathbf{u})$ and $\mathbf{v}' := p(\mathbf{v})$ have the largest, respectively the smallest, x_d-coordinate among all vertices of $P' := p(P)$.



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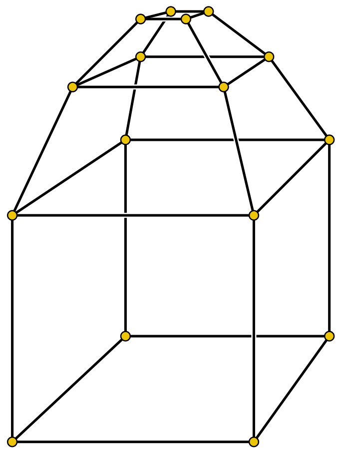
Exercise 18. **4 points**

The following “cubical stacking procedure” recycles a cubical polytope P to produce a new cubical polytope P' by “gluing” a cube onto a facet of P. More precisely: Given a cubical d-polytope P choose a facet F of P. Stack this facet by “gluing” a pyramid over F onto the facet such that the result is convex again and every vertex of P stays a vertex of the resulting polytope. (See Exercise 6, except that F is now a $(d-1)$-cube instead of a $(d-1)$-simplex.) Then “cut off” the apex of this pyramid with a hyperplane which is parallel to F and between F and the apex. The result is the *cubical stacked polytope* P'.



An *n-fold cubical-stacked polytope* is the result of starting with a cube and applying the above procedure n times.
Give coordinates for the vertices of a 3-fold cubical-stacked 4-polytope (of your choice).

Exercise 19. **4 points**

Consider the product

$$P \times Q = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \mid \mathbf{x} \in P, \mathbf{y} \in Q \right\}$$

of two polytopes P and Q.

(a) Show that the non-empty faces of $P \times Q$ are exactly the products of the faces of P with the faces of Q.

(b) Describe the face lattice $L(P \times Q)$ in terms of the face lattices $L(P)$ and $L(Q)$.

(c) Write down the f-vector of $P \times Q$ in terms of the f-vectors of P and Q.

The

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will take place at SCHLEUSENKRUG on **Wed**, **14 Mar**, starting at **18:00**.

Exercise 20. **(Tutorial)**

A *projective transformation* of $\mathbb{R}^d$ is a mapping p that maps a point $\mathbf{x} \in \mathbb{R}^d$ in the following way: Embed $\mathbf{x}$ as $\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$ into $\mathbb{R}^{d+1}$, then apply a linear transformation $T \in GL_{d+1}\mathbb{R}$; the image point $p(\mathbf{x})$ of $\mathbf{x}$ under p is the intersection of the ray $\left\{\lambda \cdot T\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \,\middle|\, \lambda \geq 0\right\}$ with the plane $\left\{\begin{pmatrix} 1 \\ \mathbf{y} \end{pmatrix} \,\middle|\, \mathbf{y} \in \mathbb{R}^d\right\} \cong \mathbb{R}^d$.

(a) Given a polytope $P \subset \mathbb{R}^d$, let the polytope $P' = p(P)$ be the image of P under a projective transformation p. Show that P' is of the form

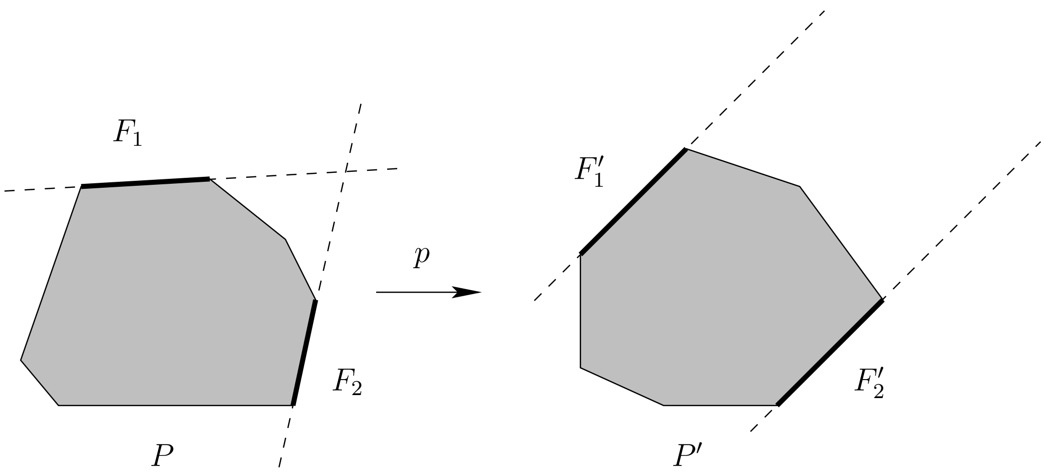
$$P' = \left\{\frac{S\mathbf{x} + \mathbf{z}}{\mathbf{a}^\top \mathbf{x} + t} \,\middle|\, \mathbf{x} \in P\right\}$$

where $t \in \mathbb{R}$, $\mathbf{a}, \mathbf{z} \in \mathbb{R}^d$ and S is a $(d \times d)$-matrix such that $\det\begin{pmatrix} t & \mathbf{a}^\top \\ \mathbf{z} & S \end{pmatrix} \neq 0$ and we have $\mathbf{a}^\top \mathbf{v} + t > 0$ for all vertices $\mathbf{v}$ of P.

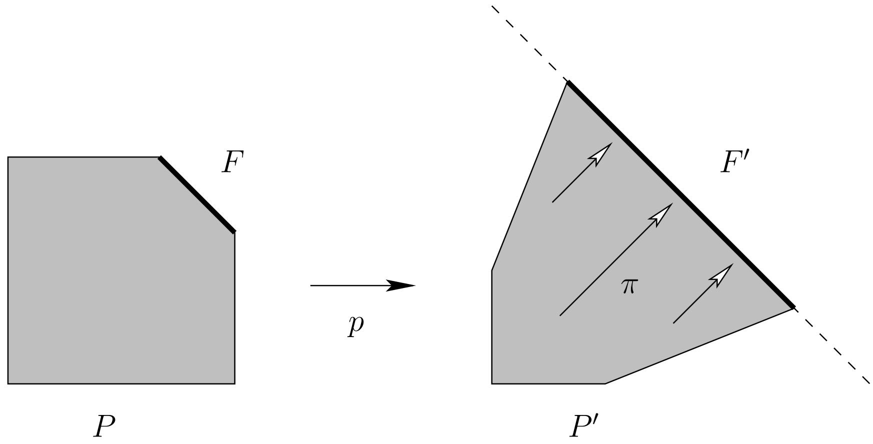
(b) Let $\mathbf{x}_1, \ldots, \mathbf{x}_{d+2}$ be points in $\mathbb{R}^d$ with the property that every set of $d+1$ of them is affinely independent. Let $\mathbf{y}_1, \ldots, \mathbf{y}_{d+2} \in \mathbb{R}^d$ be another set with this property. Show that there is a projective transformation that maps $\mathbf{x}_i$ to $\mathbf{y}_i$ for all $i \in \{1, \ldots, d+2\}$.

(c) What happens to the points $\mathbf{x}$ on the hyperplane defined by $\mathbf{a}^\top \mathbf{x} + t = 0$ under the projective transformation in (a)? How does the image of a polytope P look like if one of its vertices is on this hyperplane? What if one of its facets lies there?

(d) Let $P \in \mathbb{R}^d$ be a polytope and F_1 and F_2 two facets of P that do not intersect. Show that there is a projective transformation p such that the two facets $F_1' := p(F_1)$ and $F_2' := p(F_2)$ of $P' := p(P)$ are parallel, that is the hyperplanes $\mathsf{aff}\, F_1'$ and $\mathsf{aff}\, F_2'$ do not intersect in $\mathbb{R}^d$.

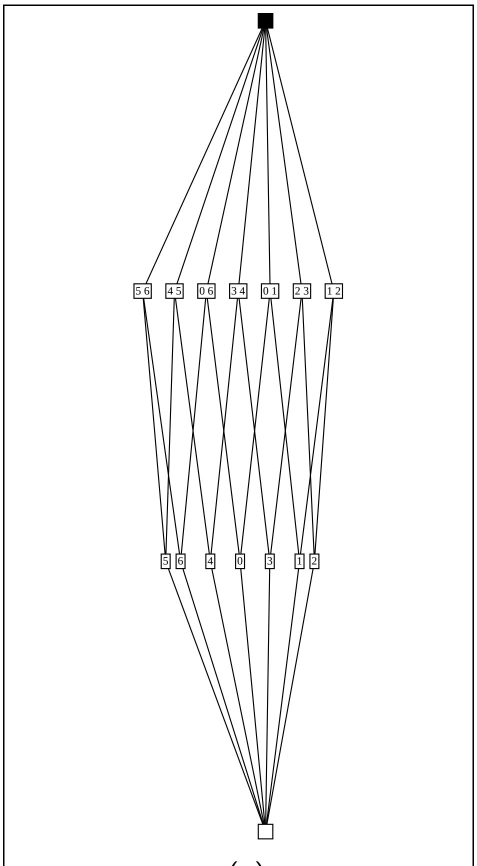


(e) Let $P \subset \mathbb{R}^d$ be a polytope and F a facet of P. Show that there is a projective transformation p with the following property: If $P' := p(P)$ and $F' := p(F)$ and $\pi : \mathbb{R}^d \to \mathsf{aff}\, F'$ is the projection map onto the hyperplane defined by F', then $\pi(P') \subseteq \pi(F')$

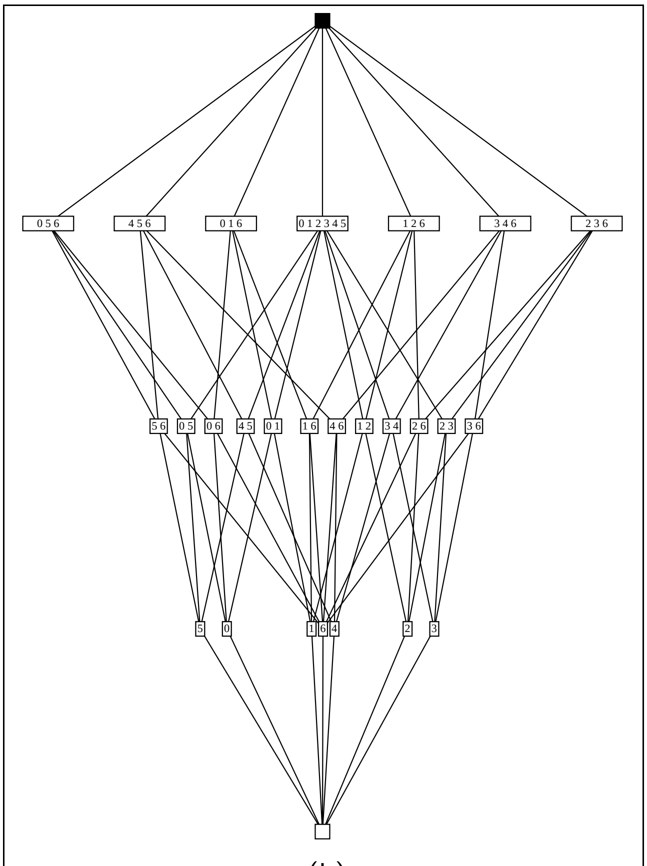


Exercise 21. **(Tutorial)**

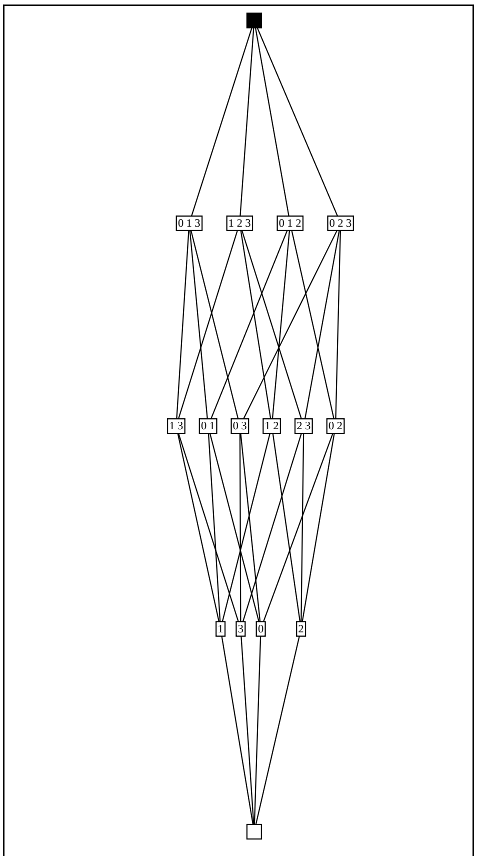
Describe the polytopes that have the following face lattices:



(a)



(b)



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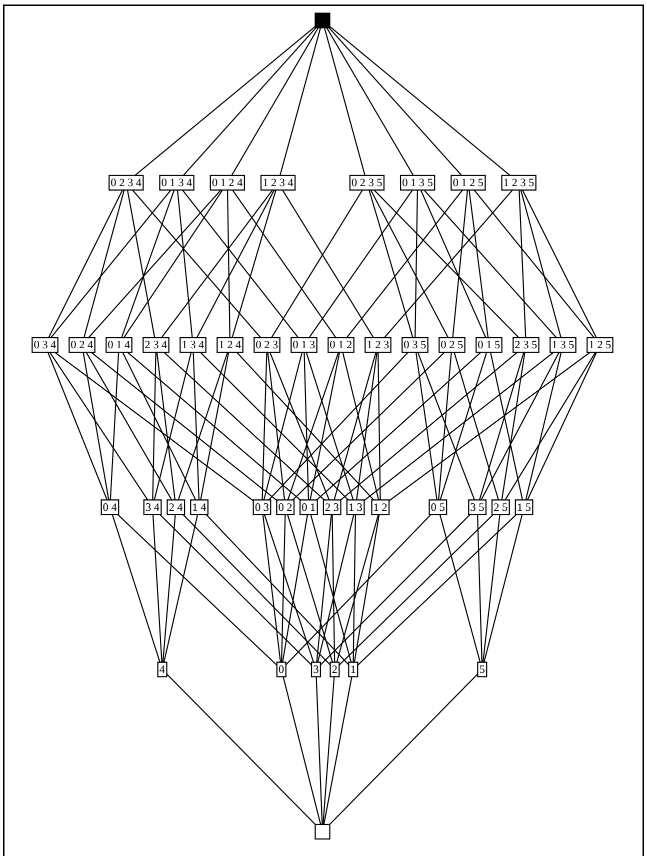

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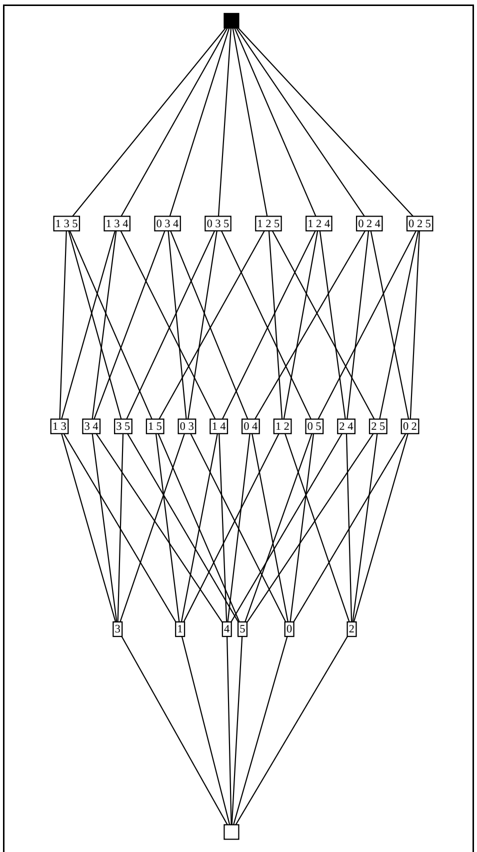
(d)



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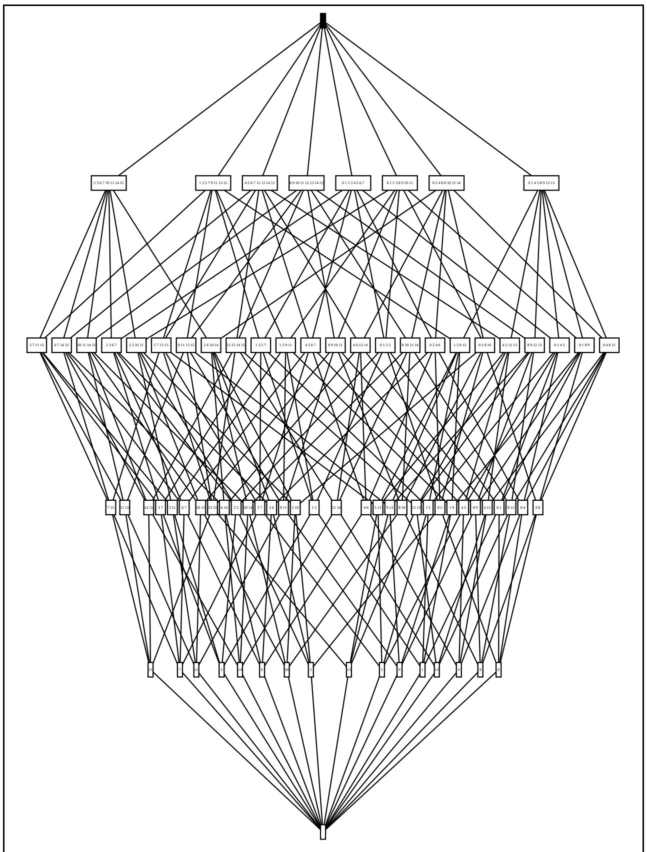
(e)



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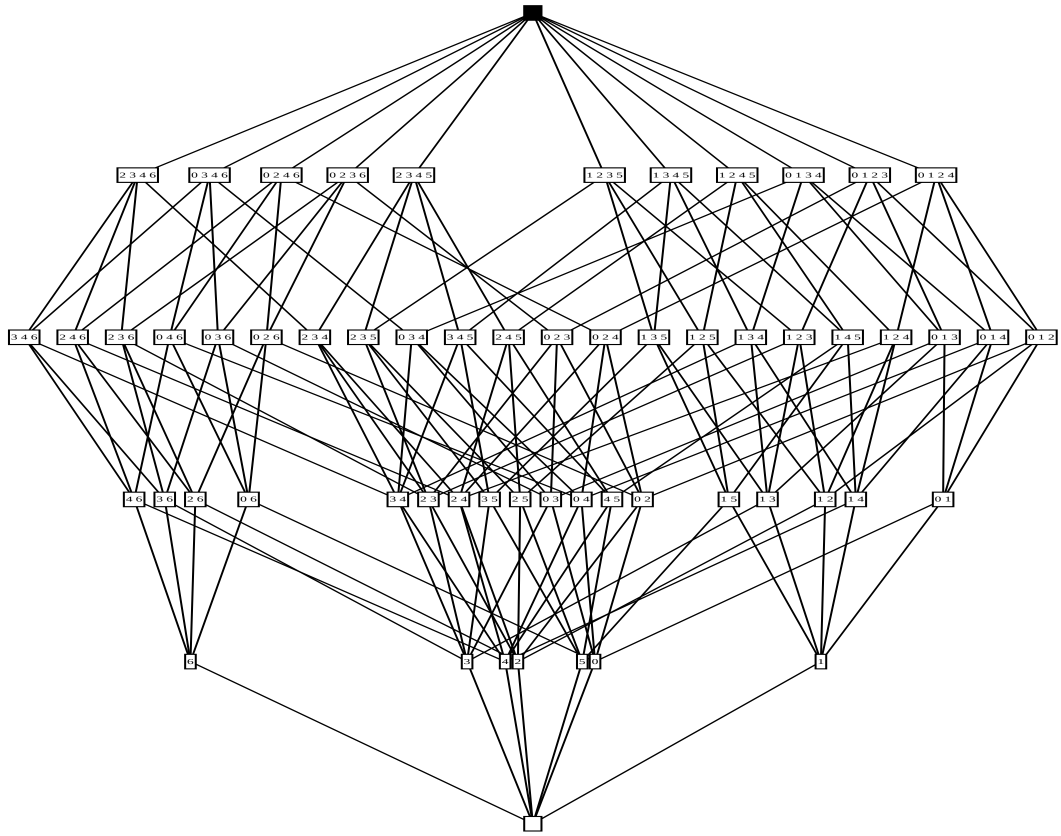
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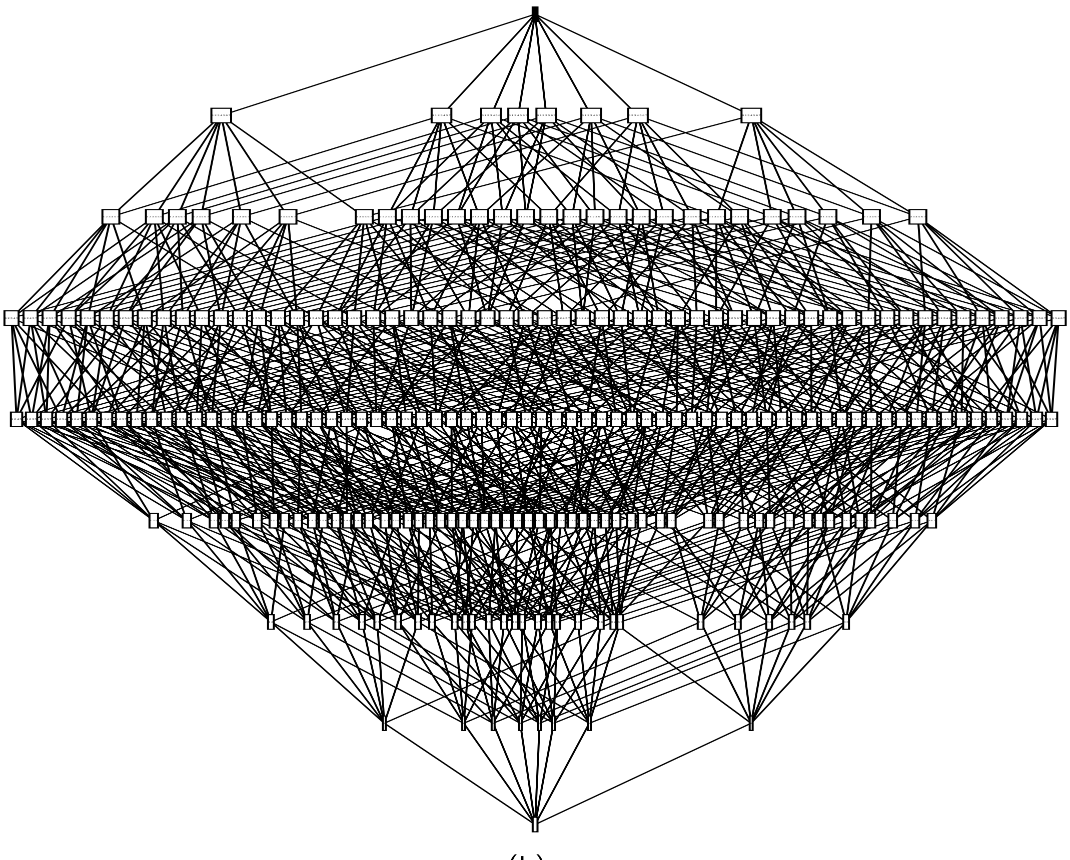
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