Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Discrete Geometry

(Kombinatorische Geometrie I)

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Exercise Sheet 7

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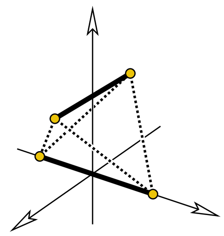
Exercise 31. **4 points**

Given two polytopes P and Q, their *join* is the polytope

$$P * Q \ := \ \mathsf{conv}\left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \\ 1 \end{pmatrix} \;\middle|\; \mathbf{x} \in P, \mathbf{y} \in Q \right\}.$$



For example, the join of two 1-polytopes $[-1,1] * [-1,1]$ is a tetrahedron.

(a) Show that the dimension of the join $P * Q$ is $\mathsf{dim}(P * Q) = \mathsf{dim}\, P + \mathsf{dim}\, Q + 1$. What is the join of a polytope P with a 0-dimensional polytope (i.e. a point)?

(b) Show that there are 5-dimensional polytopes where the number of 2-faces is quadratic in the number of vertices *and* facets: $f_2 \in \Omega((f_0 + f_4)^2)$.

Exercise 32. **4 points**

Show that the f-vectors of 2-simple, 2-simplicial 4-polytopes are symmetric.

Exercise 33. **4 points**

Consider the d-dimensional standard cube $C_d = \mathsf{conv}\{-1, 1\}^d$.

(a) Give a combinatorial description of all faces of C_d in terms of d-tuples that contain the symbols $+$, $-$ and $*$. How do you read off the dimension of a face in this notation?

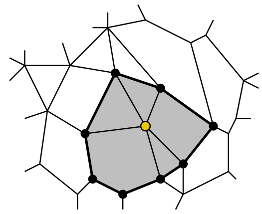
(b) Show that the f-vector of C_d is given by

$$f_k(C_d) \ = \ \binom{d}{k} \cdot 2^{d-k} \qquad \text{for } 0 \le k \le d-1.$$

What can you say about the flag vector?

Exercise 34. **4 points**

Recall the definitions of the *star* and the *link* of a vertex v in a polyhedral complex $\mathcal{C}$:



$$\begin{aligned}\mathsf{star}(v,\mathcal{C}) &:= \{F \in \mathcal{C} \mid \exists G \in \mathcal{C} : F \subseteq G \text{ and } v \in G\}\\ \mathsf{link}(v,\mathcal{C}) &:= \{F \in \mathcal{C} \mid F \in \mathsf{star}(v,\mathcal{C}) \text{ and } v \notin F\}\end{aligned}$$

Let $\mathcal{C}$ be the boundary complex of a polytope P and $\mathbf{v}$ a vertex of P. Show that the link of $\mathbf{v}$ in $\mathcal{C}$ is shellable.
Hint: Use a point beyond $\mathbf{v}$ to show that the link is combinatorially isomorphic to the boundary complex of some polytope.

Exercise 35. **(Tutorial)**

(a) Describe what happens if you "break" a facet of a simple polytope by "pulling" a vertex out of the affine hull of the facet. Make sketches for the 3-dimensional and 4-dimensional case.
How about "pushing" the vertex? What happens if the vertex is "cut off"?

(b) Find two combinatorially different 4-polytopes that have the same graph.