Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
Sommersemester 2008

Discrete Geometry

(Kombinatorische Geometrie I)

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Exercise Sheet 8

Deadline: 16 Jun 2008

Exercise 36. **4 points**

Give an example of a *pure* simplicial complex that has an h-vector with negative entries.

Exercise 37. **4 points**

Show that the d-cube C_d is extendably shellable.
(*Hint:* Characterise the pure, $(d-1)$-dimensional subcomplexes of ∂C_d that are shellable, by considering "opposite" facets.)

Exercise 38. **4 points**

Calculate the f-vector of the d-dimensional hypersimplex

$$\Delta_d(k) \;=\; [0,1]^{d+1} \cap \left\{ \mathbf{x} \in \mathbb{R}^{d+1} \;\middle|\; \sum_{i=1}^{d+1} x_i = k \right\}$$

for $2 \leq k \leq d-1$. Verify your result with Euler's equation.
(*Hint:* Use Exercise 33.)

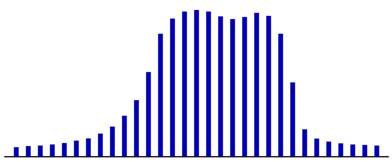
Exercise 39. **4 points**

Plot the f-vector shapes of the following polytopes (you might want to use a computer algebra program like `Mathematica` or `Maple`):

(a) the cyclic polytope $\mathcal{C}_{10}(20)$,

(b) a 10-dimensional stacked polytope with 20 vertices,

(c) the 10-dimensional hypersimplex $\Delta_{10}(5)$.



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Exercise 40. **(Tutorial)**

Given a d-polytope P and a subset $S = \{i_1, \ldots, i_\ell\} \subseteq \{0, \ldots, d-1\} =: [d]$ with $i_1 < \ldots < i_\ell$, let

$$f_S(P) := \#\{F_{i_1} \subset \ldots \subset F_{i_\ell} \mid F_{i_j} \text{ face of } P, \operatorname{dim} F_{i_j} = i_j \text{ for all } j\}$$

be the number of chains (or flags) of faces of P whose dimensions exactly match the numbers given by S.
The *(complete) flag vector* of P is the 2^d-tuple

$$\big(f_S(P)\big)_{S\subseteq[d]}$$

Obviously, the f-vector is a part of the flag vector:

$$f(P) = \big(f_S(P)\big)_{S\subseteq[d],|S|=1}$$

(a) Give the flag vectors of your favourite polytopes, for example the 3-dimensional cube and the octahedron, pyramids, stacked polytopes etc. What are the entries of the flag vector of the d-simplex? What is $f_\emptyset(P)$ for an arbitrary polytope P? What is $f_S(P^\Delta)$ in terms of the flag vector of P?

(b) Show that for 4-polytopes we only need to know the entries f_0, f_1, f_2 and f_{02} to determine the complete flag vector.

(c) Can you tell from the f-vector of a polytope whether it is simplicial/simple or not? Can you tell from its f-vector whether it is 2-simplicial? Give a criterion for a polytope to be 2-simplicial/2-simple in terms of the flag vector.

(d) (A generalisation of (b):) For a d-polytope it is enough to know the entries f_S, where $S \subseteq [d]$ is a *sparse set*, that is, $S \subseteq \{0, \ldots, d-2\}$ and S contains no two consecutive numbers.
This follows from the *Generalized Dehn-Sommerville equations*, due to Bayer and Billera (1985): if $S \subset [d]$ and we have $i, j \in S$ with $0 \leq i < j \leq d-1$ such that no integer k with $i < k < j$ is contained in S (i.e. S has a "gap" between i and j), then

$$\sum_{k=i+1}^{j-1} (-1)^{k-i-1} f_{S\cup\{k\}} \;=\; (1-(-1)^{j-i-1}) f_S.$$

The formula even makes sense if $i = -1$ or $j = d$, taking into account that $f_{-1,i_1,\ldots,i_\ell} = f_{i_1,\ldots,i_\ell} = f_{i_1,\ldots,i_\ell,d}$. Finally, for $i = -1$ *and* $j = d$ we get back Euler's equation.