Discrete Geometry

(Kombinatorische Geometrie I)

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Exercise Sheet 11

Deadline: 7 Jul 2008

Exercise 50.   **4 points**

Give two tetrahedra such that their Minkowski sum has $4 \cdot 4 = 16$ vertices.

Can you find two triangles in the plane such that their sum has $3 \cdot 3 = 9$ vertices?

If not, how many vertices can you achieve at the most?

(*Hint:* Consider the normal fans of the polytopes.)

Exercise 51.   **4 points**

Prove that the region $\mathsf{reg}(p)$ of a point p in the Voronoi diagram of a finite point set $P \subset \mathbb{R}^d$ is unbounded if and only if p lies on the boundary of $\mathsf{conv}(P)$.

Exercise 52.   **4 points**

Let $P \subset \mathbb{R}^2$ be a finite point set with no 3 points collinear and no 4 points on a circle. We define a graph on P as follows: two points p and q are connected by an edge if and only if there exists a circular disc with both p and q on the boundary and with no point of P in its interior.

Show that this graph is the Delaunay triangulation of P.

Exercise 53.   **4 points**

Consider the lifting function that gives the paraboloid:

$$f : \mathbb{R}^d \to \mathbb{R}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \mapsto \sum_{i=1}^{d} x_i^2$$

Let $P \subset \mathbb{R}^d$ be a finite point set. We consider the regular subdivision $\mathcal{C}$ of $\mathsf{conv}(P)$ that is obtained by taking the convex hull of the lifted points

$$\left\{ \begin{pmatrix} \mathbf{p} \\ f(\mathbf{p}) \end{pmatrix} \mid \mathbf{p} \in P \right\}$$

and projecting its lower envelope down to $\mathbb{R}^d$ again.

Show that for every facet F of $\mathcal{C}$ there is a d-ball B such that the vertices of F are all on the boundary of B and its interior contains no points from P.

Exercise 54. **(Tutorial)**

(a) Consider the set of 2-dimensional vectors

$$V = (\mathbf{v}_1, \ldots, \mathbf{v}_5) = \begin{pmatrix} 4 & 0 & -1 & -1 & 1 \\ 0 & 2 & 2 & -4 & -3 \end{pmatrix}$$

- Construct the zonotope $Z(V)$ and label the faces with their sign vectors.
- Relate the cells of the arrangement $\mathcal{A}_V$ to the faces of $Z(V)$.
- Construct $Z(V)^\triangle$ and show that its face fan coincides with the fan of $\mathcal{A}_V$.

(b) Try to understand the zonotope and arrangement for the vectors

$$V = (\mathbf{v}_1, \ldots, \mathbf{v}_5) = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Construct an example of a zonotope that is not cubical. Construct one that is neither cubical nor simple.