Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
Sommersemester 2008

Discrete Geometry

(Kombinatorische Geometrie I)

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Exercise Sheet 12

(last problem set of semester)

Deadline: 14 Jul 2008

Exercise 55. **4 points**

For a point $p = (p_x, p_y)$ in the plane we define its *dual line*

$$p^* := \{(x, y) \in \mathbb{R}^2 \mid y = p_x x - p_y\}.$$

Accordingly, for a non-vertical line $\ell = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$ in the plane, the *dual point* is defined by $\ell^* := (m, -b)$. (Note that there are no dual points of vertical lines!) Obviously, we have $(p^*)^* = p$ for all points p and $(\ell^*)^* = \ell$ for all non-vertical lines ℓ.

(a) Sketch the dual line configuration for the point set

$$\{(0, 0), (1, 1), (1, -1), (-2, 1), (-1, -1)\}$$

(b) Show that this concept of duality is *incidence preserving* and *order preserving*: For every point p and non-vertical line ℓ

- $p \in \ell$ if and only if $\ell^* \in p^*$ and
- p lies above ℓ if and only if ℓ^* lies above p^*.

(c) Show that a finite point set P is in general position if and only if the line arrangement $P^* = \{p^* \mid p \in P\}$ is simple.

(d) Describe the dual of a line segment $[p, q]$ between two points p and q and the dual of the convex hull of a finite set of points.

Exercise 56. **4 points**

Calculate the number of k-faces of a simple arrangement of n hyperplanes in $\mathbb{R}^d$.

Exercise 57. **4 points**

(a) Prove that an arrangement of d of fewer hyperplanes in $\mathbb{R}^d$ has no bounded cell.

(b) Prove that an arrangement of $d + 1$ hyperplanes in general position in $\mathbb{R}^d$ has exactly one bounded cell.

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Exercise 58. **4 points**

How many d-dimensional cells does the (non-simple) arrangement

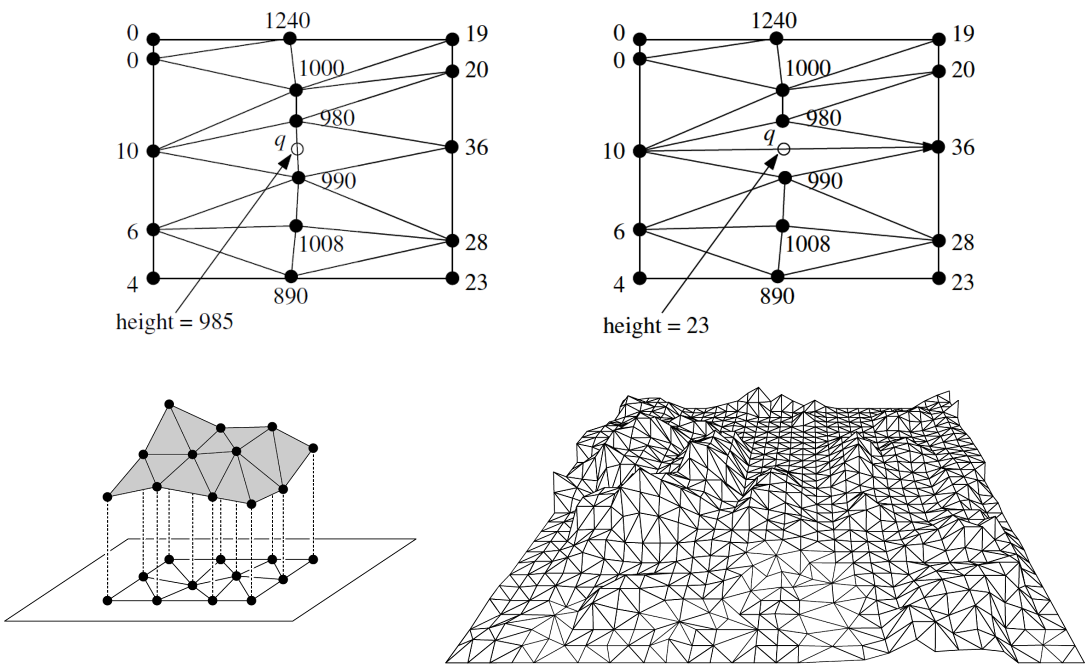
$$H = \Big\{ \{ \mathbf{x} \in \mathbb{R}^d \mid x_i = x_j \} \;\Big|\; 1 \le i < j \le d \Big\}$$

of $\binom{d}{2}$ hyperplanes in $\mathbb{R}^d$ have? Identify the zonotope associated with this arrangement.

Exercise 59. **(Tutorial)**

(a) Show that the dual of the Voronoi diagram of a finite set of points $P \subset \mathbb{R}^d$ defines a polytopal subdivision of $\mathsf{conv}\, P$. When is this subdivision a triangulation?

(b) Show that the Delaunay triangulation of a point set in the plane is *angle-optimal* (that is, it maximises the angles in the triangles of the subdivision) by starting with an arbitrary triangulation and *flipping* edges.



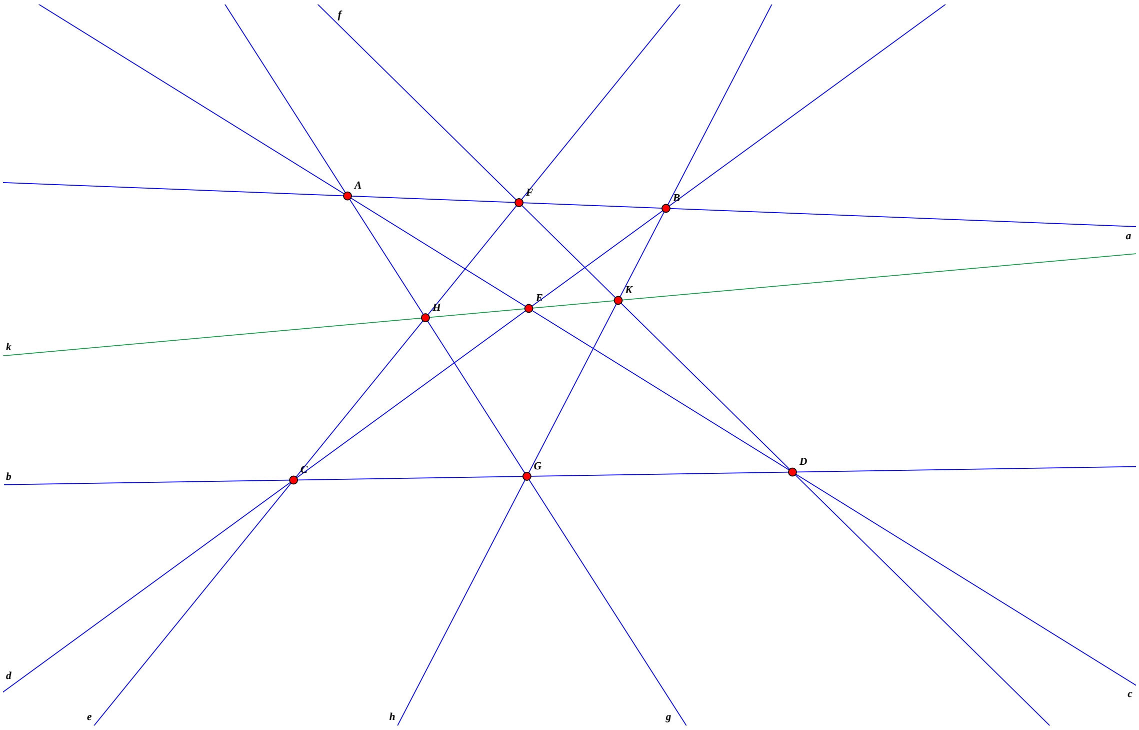
From: M. de Berg et al., *Computational Geometry*, Springer, 2000

The

☆ FINAL EXAM ☆

will take place on **Fri**, **1 Aug 2008**, **10:00–12:00** in room **MA 841**.

Pappus' theorem:



Desargues' theorem: