Technische Universität Berlin
Institut für Mathematik
J. Behrndt, C. Kreusler

Summer semester 2008

Functional Analysis I

11th problem sheet

Please return your answers in the tutorials on July, 3rd / 4th.

Problem 1: **4 pt.**

Let $X = C([0,1])$, equipped with the maximum norm, and let $k \in C([0,1]^2)$. Show that the operator

$$T : X \to X, \quad (Tu)(t) = \int_0^1 k(s,t)u(s)ds, \ t \in [0,1], u \in X$$

is compact.

Problem 2: **6 pt.**

Let X be a normed space and let $U \subset X$ and $V \subset X'$ be subspaces.

(i) Show that both $U^\perp$ and $V_\perp$ are closed subspaces.

(ii) Assume that U is closed. Show that

$$T : (X/U)' \to U^\perp, \quad f \mapsto f \circ q$$

is a well defined isomorphism where $q : X \to X/U$ is the canonical quotient mapping given by$q(x) = [x]$ for $x \in X$.

(iii) Assume that U is closed. Show $U' \cong X'/U^\perp$.

Problem 3: **5 pt.**

Ley X and Y be Banach spaces and let $\in L(X,Y)$ be a Fredholm operator, that is ran T is closed and both $\ker T$ and $Y/\text{ran } T$ are finite dimensional[1]. Show that the adjoint operator T' is also a Fredholm operator.

Problem 4: **5 pt.**

Let X and Y be normed spaces and $T : X \to Y$ be linear. Assume that ran T is finite dimensional. Show that there exist linearly independent $y_1, \dots, y_n \in Y$ and $f_1, \dots, f_n \in X'$ such that

$$Tx = \sum_{x=1}^{n} f_i(x) y_i \quad \text{for all } x \in X.$$

[1] In fact, one may show that the closedness of ran T already is implied by the two other conditions. The standard examples of Fredholm operators are of the form $T = I - S$ where S is compact.