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Summer semester 2008

Functional Analysis I

2nd problem sheet

Please return your answers in the exercise on April, 30th,
or in the tutorium on May, 2nd.
(Remember that there're no lectures / tutorials on May, 1st.)

Problem 1: **4 pt.**

Let $p \in [1, \infty)$ and consider the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$ on $C([0,1])$.

(i) Find a sequence (f_n) in $C([0,1])$ and a number $M > 0$ such that $\|f_n\|_p \to 0$ as $n \to \infty$ and $\|f_n\|_\infty \geq M$ for all $n \in \mathbb{N}$. Show that $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are not equivalent on $C([0,1])$.

(ii) Find a sequence (f_n) in $C([0,1])$ which is bounded by one with respect to the maximum norm and which does not contain any convergent subsequence (this shows that the unit ball in $(C([0,1], \|\cdot\|_\infty)$ is not compact).

Problem 2: **4 pt.**

Show that for all $u \in l^1(\mathbb{N})$

$$\lim_{p \to \infty} \|u\|_p = \|u\|_\infty.$$

Problem 3: **6 pt.**

A normed space $(X, \|\cdot\|)$ is called **strictly convex** if $\|x + y\| < 2$ holds for all $x, y \in X$, $\|x\| = \|y\| = 1$, $x \neq y$. (What does the unit ball looks like in this case?)
Show that for $1 < p < \infty$ the spaces $l^p(\mathbb{N})$ are strictly convex, but $l^1(\mathbb{N})$ and $l^\infty(\mathbb{N})$ are not.

Problem 4: **6 pt.**

Let $\Omega \subset \mathbb{R}^n$ (be measurable) and $1 \leq p \leq q \leq \infty$. Show that $L^q(\Omega) \subset L^p(\Omega)$ holds if and only if Ω has bounded measure. Show that in this case

$$\|f\|_p \leq c\|f\|_q, \quad f \in L^q(\Omega)$$

for some constant c which does not depend on f. We then call $L^q(\Omega)$ **continuously embedded** in $L^p(\Omega)$ which is often denoted by $L^q(\Omega) \hookrightarrow L^p(\Omega)$.
Does $L^p(\Omega) \subset L^q(\Omega)$ hold if Ω has bounded (unbounded, resp.) measure?