Technische Universität Berlin
Institut für Mathematik
J. Behrndt, C. Kreusler

Summer semester 2008

Functional Analysis I

3rd problem sheet

Please return your answers in the tutorials on May, 8th / 9th.

Problem 1: **4 pt.**

Let $1 \le p < \infty$, $(f_n) \subset L^p([0,1])$ and $f : [0,1] \to \mathbb{R}$. Prove or disprove the following propositions.

(i) If (f_n) converges to f pointwisely, then it converges to f in $L^p([0,1])$, i.e. $\|f_n - f\|_p \to 0$ as $n \to \infty$.

(ii) If (f_n) converges to f uniformly, then it converges to f in $L^p([0,1])$.

(iii) If (f_n) converges to f in $L^p([0,1])$, then it converges to f uniformly.

Problem 2: **4 pt.**

Let X be a vector space and $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ two norms on X such that both $(X, \|\cdot\|)$ and $(X, |\!|\!|\cdot|\!|\!|)$ are complete. Furthermore assume that there is an $M > 0$ such that

$$\|x\| \le M |\!|\!|x|\!|\!|, \quad x \in X.$$

Show that $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent.

Problem 3: **4 pt.**

Show that the *Hilbert cube*

$$H := \{x \in l^2(\mathbb{N}) \mid |x_n| \le 1/n,\ n \in \mathbb{N}\}$$

is a compact subset of $l^2(\mathbb{N})$.

Problem 4: **5 pt.**

Let $(X, \|\cdot\|)$ be a Banach space. A subset $A \subset X$ is called *relatively compact* if the closure $\overline{A}$ of A is compact. A is called *totally bounded* if for every $\epsilon > 0$ there are finitely many $x_1, \dots, x_N \in A$ such that

$$A \subset \bigcup_{i=1}^{N} B(x_i, \epsilon).$$

Here for $x \in X$, $r > 0$ the set $B(x,r) := \{y \in X \mid \|y - x\| < r\}$ denotes the open ball centered in x with radius r.
Show that a subset $A \subset X$ is relatively compact if and only if A is totally bounded.

Problem 5: **3 pt.**

Let $(X, \|\cdot\|)$ be a normed space. Show that the following are equivalent:

(i) X is separable.

(ii) $\overline{B}(0,1) := \{x \in X \mid \|x\| \leq 1\}$ is separable.

(iii) $S := \{x \in X \mid \|x\| = 1\}$ is separable.