Tutorials on April, 17th / 18th

Problem 1:
Let (X, d) be a metric space. Show that

$$\tilde{d}(x,y) := \frac{d(x,y)}{1+d(x,y)}, \quad x, y \in X,$$

provides another metric on X.

Problem 2:
Define a family of mappings $p_t : C([-1,1]) \to [0,\infty)$ with $t \in [-1,1]$ via

$$p_t(f) := |f(t)|, \quad f \in C([-1,1]).$$

Show that p_t is a semi-norm on $C([-1,1])$ for every $t \in [-1,1]$. Why is it not a norm?

Problem 3:
Let $A \subset \mathbb{R}^d$, $\lambda \in (0,1]$ and $C^\lambda(A)$ be the class of Hölder continuous functions. Show $C^\lambda(A) \subset C(A)$, i.e. every Hölder continuous function is continuous.

Problem 4:
Let X be a vector space and $p : X \to [0,\infty)$ be a mapping with

$$\begin{aligned} (i)&\ \ p(x) = 0 \iff x = 0 \\ (ii)&\ \ p(\lambda x) = |\lambda| p(x) \quad \text{for all } \lambda \in \mathbb{R},\ x \in X. \end{aligned}$$

Show that p is a norm on X if and only if the closed unit ball with respect to p is convex.

Problem 5:
Let E be a normed space. Assume that every sum $\sum_{n=0}^\infty u_n$ converges if $\sum_{n=0}^\infty \|u_n\| < \infty$. Show that E is a Banach space.
Show that the opposite direction is also true.
Hint: Recall that a Cauchy sequence converges if it has a convergent subsequence. (Why?)