Tutorium on May, 2nd

Problem 1:
Let (X,d), $(\hat{X},\hat{d})$ be a metric spaces, $A \subset X$ compact and $f : X \to \hat{X}$ continuous. Show that $f(A)$ is compact.

Problem 2:
For $f \in C^1[0,1]$ we define the expressions

$$\begin{aligned}
|||f|||_1 &:= |f(0)| + \|f'\|_\infty, \\
|||f|||_2 &:= \max\{|f(0)|, \|f'\|_\infty\}, \\
|||f|||_3 &:= \Big(\int_0^1 |f(t)|^2 dt + \int_0^1 |f'(t)|^2 dt \Big)^{1/2}. \\
|||f||| &:= \|f\|_\infty + \|f'\|_\infty.
\end{aligned}$$

Convice yourself that all these expressions define norms on $C[0,1]$.
Which of them are (pairwise) equivalent?

Problem 3:
Let (X,d) be a metric space and $A \subset X$ a closed subset. For $x \in X$ we define

$$d(x,A) := \inf_{y \in A} d(x,y).$$

(i) Show that $d(\cdot,A) : X \to \mathbb{R}$ is continuous and

$$A = \{x \in X \mid d(x,A) = 0\}.$$

(ii) Assume X to be finite dimensional. Show that $\delta = 0$ may be allowed in the Riesz Lemma (in contrast to the infinite dimesional case).

Problem 4:
Let (X,d) be a normed space and let $Y \subset X$ be a linear subspace. For a given $x \in X$ the *best approximation* in Y is an element $\hat{y} \in Y$ such that

$$\|x - \hat{y}\| = d(x,Y) = \inf_{y \in Y} \|x - y\|.$$

(i) Show that best approximations need not be unique in general (but if $(X, \|\cdot\|)$ is strictly convex, uniqueness may be proven).

(ii) Show that best approximations need not exist: Consider $X = C([0,1/2])$ and let Y be the subspace of all polynomials. Show there is no best approximation to $f \in X$, $f(t) = 1/(1-t)$, $t \in [0,1/2]$.