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Integer Linear Programming
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Assignment 2

Exercise 1

Show that there are lattices that have no orthogonal basis.

Exercise 2

We want to show that Lovász's basis reduction method runs in polynomial time. Therefore we define the potential

$$\Phi(b_1, \ldots, b_n) = \prod_{i=1}^{n} \det(B_i^T D B_i)$$

where $B_i = (b_1, \ldots, b_i)$.

a) Show that $\Phi(b_1, \ldots, b_n)$ does not change in steps (1) and (2) of the algorithm.

b) Show that $\det(B_i^T D B_i) = \prod_{j=1}^{i} \|b_j^*\|_D^2$, where $b_1^*, \ldots, b_i^*$ is the Gram-Schmidt orthogonalization of $b_1, \ldots, b_i$ with respect to D.

c) Suppose b_k and b_{k+1} are interchanged in step (3) for one k (Reminder: This is the case if $\|b_k^*\|_D^2 > 2\|b_{k+1}^*\|_D^2$). Use b) to show that then $\det(\tilde{B}_k^T D \tilde{B}_k) < \frac{3}{4} \det(B_k^T D B_k)$, where $\tilde{B}_k$ arises from B_k by replacing its kth column with b_{k+1}.

d) Use c) to show that $\Phi(b_1, \ldots, b_n)$ decreases by a factor of $\frac{3}{4}$ after every execution of step (3).

e) Show that initially $\Phi(b_1, \ldots, b_n) \leq (nd_{\max})^{n^2}$, where $d_{\max} := \max_{i,j} |d_{ij}|$. Show that $\Phi(b_1, \ldots, b_n) \geq 0$ throughout the algorithm.

f) Conclude that Lovász's basis reduction method runs in polynomial time.

Exercise 3

Apply Lovász's basis reduction method to the lattice generated by the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Exercise 4

Show that Integer Linear Programming does not yield strong duality, i.e. find a matrix $A \in \mathbb{Q}^{m\times n}$ and vectors $c \in \mathbb{Q}^n, b \in \mathbb{Q}^m$ such that

$$\begin{aligned} \max\{c^T x \mid Ax \leq b,\ x \in \mathbb{Z}^n\} &< \max\{c^T x \mid Ax \leq b,\ x \in \mathbb{Q}^n\} \\ &= \min\{b^T y \mid A^T y = c,\ y \geq 0,\ y \in \mathbb{Q}^m\} \\ &< \min\{b^T y \mid A^T y = c,\ y \geq 0,\ y \in \mathbb{Z}^m\}. \end{aligned}$$

What can you say about complementary slackness?