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Integer Linear Programming
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Assignment 3

Exercise 1

We want to show that in Lenstra's algorithm, the task of finding an integral point of $P_t := P \cap \{x \in \mathbb{R}^n \mid c^T x = t\}$, with $c \in \mathbb{Z}^n$ and $t \in \mathbb{Z}$, can indeed be reduced to that of finding an integral point in a polyhedron in $\mathbb{R}^{n-1}$.

a) Show that we can assume w.l.o.g. that $\gcd(\{c_1, \ldots, c_n\}) = 1$.

b) Find a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ and a polyhedron $\tilde{Q}_t \subseteq \{x \in \mathbb{R}^n \mid x_1 = t\}$ such that $x \in P_t$ iff $Ux \in \tilde{Q}_t$.

c) Let Q_t be the projection of $\tilde{Q}_t$ to the last $n-1$ components. Show how to construct an integral point of Q_t out of an integral point of P_t and vice versa.

Exercise 2

We want to show that a cone is finitely generated iff it is polyhedral. For this reason we will prove the

Fundamental Theorem of Linear Inequalities: Let $a_1, \ldots, a_m \in \mathbb{R}^n$, with $\operatorname{rank}(a_1, \ldots, a_m) = n$. Then for every $b \in \mathbb{R}^n$ one of the following statements is true.

(A) $b \in \operatorname{cone}(a_1, \ldots, a_m)$

(B) There exists a hyperplane $\{x \in \mathbb{R}^n \mid c^T x = 0\}$ that separates b from $\{a_1, \ldots, a_m\}$ (i.e., $c^T b < 0$ and $c^T a_i \geq 0$ for all $i \in \{1, \ldots, m\}$) and that also contains at least $n-1$ linearly independent vectors of $a_1, \ldots, a_m$.

In order to prove the theorem consider the following algorithm.

Step 0 Choose $D \subseteq \{1, \ldots, m\}$ with $|D| = n$ such that $\{a_i | i \in D\}$ is linearly independent.

Step 1 Let $b = \sum_{i \in D} \lambda_i a_i$ be the unique representation of b w.r.t. $\{a_i \mid i \in D\}$. If $\lambda_i \geq 0 \ \forall i$ terminate. Otherwise let $k := \min\{i \in D \mid \lambda_i < 0\}$.

Step 2 Let $c \in \mathbb{R}^n$ be the unique solution to $c^T A_D = e_k^T$. If $c^T a_i \geq 0$ for all $i \in \{1, \ldots, m\}$ terminate. Otherwise let $l := \min\{i \in \{1, \ldots, m\} \mid c^T a_i < 0\}$, set $D := (D \setminus \{k\}) \cup \{l\}$ and goto Step 1.

a) Show that if the algorithm terminates, either (A) or (B) is fulfilled.

b) Show that the algorithm always terminates and hence the fundamental theorem is correct.

c) Show that every fulldimensional finitely generated cone is polyhedral.

d) Show that every finitely generated cone is polyhedral.

e) Prove the converse. *Hint: Use the technique from the lecture.*

Exercise 3

Find a system of 2^n linear inequalities in n variables that has no integral solution, but whenever we drop a single inequality there is an integral solution fulfilling the $2^n - 1$ inequalties that are left.

Exercise 4

Consider the polyhedron $P := \{(x, y) \in \mathbb{R}^2 \mid y \leq \sqrt{2}x\}$. Show that its integer hull $P_I :=$ conv. hull$(P \cap \mathbb{Z}^2)$ is not a polyhedron, i.e., it cannot be described by finitely many inequalities.

Exercise 5

Find a cone that does not have a unique inclusionwise minimal Hilbert basis.