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Integer Linear Programming
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Assignment 4

Exercise 1

Consider the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Prove or disprove: For any given $b \in \mathbb{Z}^n$ the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is integral.

Exercise 2

Prove Cramer's Rule: For a regular matrix $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ the unique solution to $Ax = b$ is given by

$$x_i = \frac{\det(\tilde{A}_i)}{\det(A)}$$

where $\tilde{A}_i$ arises from A by replacing the ith column with b.

Exercise 3

Let $G = (V, E)$ be a graph. A vertex cover is a set $S \subseteq V$ such that every edge $e \in E$ has at least one endpoint in S. Show that in bipartite graphs the cardinality of a minimum vertex cover equals the cardinality of a maximum matching.

Exercise 4

Prove the total unimodularity criterion of Baum and Trotter: $A \in \mathbb{R}^{m \times n}$ is totally unimodular if and only if for every $b \in \mathbb{R}^m$, every $k \in \mathbb{N}$ and every nonnegative integral $y \in \mathbb{Z}^n_+$ with $Ay \leq kb$ there are k nonnegative integral vectors $x_1, \ldots, x_k \in \mathbb{Z}^n_+$ with $Ax_i \leq b$ for $i \in \{1, \ldots k\}$ such that $y = x_1 + \ldots + x_k$.

Exercise 5

An intervall matrix, or consecutive-0-1 matrix, is a matrix $A \in \{0,1\}^{m \times n}$ such that for all $j \in [n]$ and all $i_1, i_2 \in [m]$ with $A_{i_1 j} = A_{i_2 j} = 1$, we have $A_{kj} = 1$ for all k with $j_1 \leq k \leq j_2$, i.e. in each column the 1-entries occur consecutively. Show in two different ways that every intervall matrix is totally unimodular.

Exercise 6

For $k \in \mathbb{N}$ let $A = \begin{pmatrix} 1 & 0 \\ 1 & k \end{pmatrix}$ and consider the polyhedron $P = \left\{x \in \mathbb{R}^2 \mid Ax \leq 0\right\}$.

a) Compute a minimal Hilbert basis for the cone spanned by the rows of A.

b) Show that any TDI-system $A'x \leq 0$ that describes P has size exponential in the encoding size of A.