Technische Universität Berlin **Sommersemester 2012**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
Tobias Brüll

Systems and control theory

Series 10

Task 1:
The goal of this task is to give a result similar to Theorem 4.3 and Theorem 4.4 for matrices $P \in \mathbb{C}^{p,q}$ which are not polynomials, $H = H^* \in \mathbb{C}^{q,q}$, and vectors $z \in \mathbb{C}^q$ which do not depend on time. We therefore call H *non-negative* w.r.t. P if $0 \le y^* H y$ for all $y \in \operatorname{kernel}(P)$.
Let $\hat{z} \in \mathbb{C}^q$, let $b \in \mathbb{C}^p$. Show that then the following are equivalent:

1. We have that H is non-negative w.r.t. to P and there exists a $\hat{\mu} \in \mathbb{C}^p$ such that
$$\begin{bmatrix} 0 & P \\ P^* & H \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$
2. We have that $\hat{z}$ solves the optimization problem
$$\hat{z}^* H \hat{z} = \inf_{Pz=b} z^* H z.$$

Hint: Adapt the proves of the mentioned Theorems.

Task 2:
Let the polynomial $N(\lambda) = \sum_{i=0}^{K} \lambda^i N_i \in \mathbb{C}[\lambda]^{n,n}$ be para-Hermitian $N = N^\sim$. Show that the coefficients are Hermitian $N_i = N_i$ for even i and skew-Hermitian $N_i = -N_i^*$ for odd i. Also see Series 9, Task 2.

Task 3: Let $M \in \mathbb{C}^{n,n}$.
Show that if $M = M^*$ is Hermitian we have that $x^* M x \in \mathbb{R}$ is real for all $x \in \mathbb{C}^n$.
Show that if $M = -M^*$ is skew-Hermitian we have that $x^* N x \in \imath\mathbb{R}$ is imaginary for all $x \in \mathbb{C}^n$.

Task 4:
Show that for $f : \mathbb{R} \to \mathbb{C}$ we have $\int_{t_0}^{t_1} \operatorname{Re}\{f(t)\}\, dt = \operatorname{Re}\left\{\int_{t_0}^{t_1} f(t) dt\right\}$ for all $t_0 < t_1$.

Task 5:
Let $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $Q = Q^* \in \mathbb{C}^{n,n}$, $S \in \mathbb{C}^{n,m}$, and $R = R^* \in \mathbb{C}^{m,m}$ be such that $0 \le \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$.
Let $x_0 \in \mathbb{C}^n$. Show that then $(\hat{x}, \hat{u}) \in \mathcal{B}_+(\begin{bmatrix} \lambda I - A & -B \end{bmatrix}) =: \mathcal{B}_+(\lambda F + G)$ is a solution of the optimal control problem
$$\inf_{\substack{(x,u) \in \mathcal{B}_+(\lambda F + G) \\ x(0) = x_0}} \int_0^\infty x^*(t) Q x(t) + 2\operatorname{Re}\left(x^*(t) S u(t)\right) + u^*(t) R u(t) dt$$
if and only if there exists a $\hat{\lambda} \in \mathcal{C}_+^n$ such that
$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{\lambda}}(t) \\ \dot{\hat{u}}(t) \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & -A^* & -S \\ S^* & B^* & R \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{\lambda}(t) \\ \hat{u}(t) \end{bmatrix} \qquad \text{for all } t \ge 0.$$

Hint: Use Theorem 4.3, Theorem 4.4, and Theorem 4.5.

Task 6:
Let $V \in \mathbb{C}^{2p,2p}$ be unitary and set $J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Show that then $V^*JV = J$ if and only if there exist $V_1, V_2 \in \mathbb{C}^{p,p}$ such that $V = \begin{bmatrix} V_1 & V_2 \\ -V_2 & V_1 \end{bmatrix}$.

Task 7: (Complex Householder reflections)
Let $x \in \mathbb{C}^p \setminus \{0\}$ and write the last entry of x in the form $x_p = re^{\imath\omega}$ with $r, \omega \in \mathbb{R}$. Define $u := x + e^{\imath\omega}\|x\|_2 e_p$, where e_p denotes the last unit vector and $\|\cdot\|_2$ denotes the euclidian norm. Show that $P := I - 2\frac{uu^*}{u^*u}$ is unitary and that $Px = -e^{\imath\omega}\|x\|_2 e_p$.

Task 8:
Show that the assumptions "non-imaginary zero" in Lemma 4.6 and Theorem 4.7 is really necessary. Therefore, proceed along the following lines:
With a real number $a \in \mathbb{R}$ consider the para-Hermitian polynomial matrix

$$N_a(\lambda) := \lambda \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:J} + \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}.$$

(a) Give the zeros $\lambda_1, \lambda_2 \in \mathfrak{Z}(N_a)$ in dependence of a. Visualize the result graphically. For one of the zeros (say λ_1) give an $x_1 \in \mathbb{C}^2 \setminus \{0\}$ such that $N(\lambda_1)x_1 = 0$.

(b) For $a < 0$ construct a unitary matrix $V \in \mathbb{C}^{2,2}$ with $V^*JV = J$ such that with x_1 from (a) we have $Vx_1 = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$ with $\beta \in \mathbb{R}$.

(c) For $a > 0$ show that there does not exist a unitary matrix $V \in \mathbb{C}^{2,2}$ with $V^*JV = J$ such that with x_1 from (a) we have $V^*x_1 = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$ with $\beta \in \mathbb{C}$.

(d) Show that for $a > 0$ there does not exist a unitary matrix $V \in \mathbb{C}^{2,2}$ with $V^*JV = J$ such that

$$V^*N(\lambda)V = \lambda J + \begin{bmatrix} m_1 & m_2 \\ \overline{m_2} & 0 \end{bmatrix},$$

with $m_1 \in \mathbb{R}$ and $m_2 \in \mathbb{C}$, i.e., that in this case one cannot transform N to the para-Hermitian Schurform of Theorem 4.7.

Hint: For (c) & (d) use Task 6.

Task 9:
Complete the proof of Theorem 4.7:

1. Verify the base case.
2. Let $M_{11} = M_{11}^*, M_{12}, M_{22} = M_{22}^* \in \mathbb{C}^{p-1,p-1}$ and $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) \neq 0$. Show that

$$\begin{aligned} &\mathfrak{Z}\left(\lambda \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix} + \begin{bmatrix} M_{11} & \times & M_{12} & 0 \\ \times & \times & \times & -\lambda_0 \\ M_{12}^* & \times & M_{22} & 0 \\ 0 & -\overline{\lambda_0} & 0 & 0 \end{bmatrix}\right) \\ = \; &\mathfrak{Z}\left(\lambda \begin{bmatrix} 0 & I_{p-1} \\ -I_{p-1} & 0 \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}\right) \cup \left\{\lambda_0, -\overline{\lambda_0}\right\}, \end{aligned}$$

where the $\times$-symbols denote arbitrary scalars or vectors of matching dimensions.