Series 5:

Task 1:

With $P(\lambda) = \sum_{i=0}^{k} \lambda^i P_i$ we have

$$P(\tfrac{d}{dt})\, y(t) = \sum_{i=0}^{k} (\tfrac{d}{dt})^i (P_i e^{\lambda_0 t} \alpha_0)$$
$$= \sum_{i=0}^{k} P_i (\lambda_0)^i e^{\lambda_0 t} \alpha_0$$
$$= P(\lambda_0) e^{\lambda_0 t} \alpha_0 = P(\lambda_0)\, y(t).$$

Task 2:

1.) Let $\mathfrak{B}(P)$ be controllable from t_0 to t_1. To show that $\mathfrak{B}(P)$ is controllable from 0 to $t_1 - t_0$ let $z_0, z_1 \in \mathfrak{B}(P)$. Since then also

$$z_0(\cdot - t_0),\ z_1(\cdot - t_0) \in \mathfrak{B}(P)$$

(the system is time-invariant) there exists a $\tilde{z} \in \mathfrak{B}(P)$ with

$$\tilde{z}(t) = \begin{cases} z_0(t - t_0) & , t \le t_0 \\ z_1(t - t_0) & , t \ge t_1 \end{cases}.$$

Then also $z(\cdot) := \tilde{z}(\cdot + t_0) \in \mathfrak{B}(P)$ and z fulfills

$$z(t) = \tilde{z}(t + t_0) = \begin{cases} z_0(t + t_0 - t_0) & , t + t_0 \le t_0 \\ z_1(t + t_0 - t_0) & , t + t_0 \ge t_1 \end{cases} = \begin{cases} z_0(t) & , t \le 0 \\ z_1(t) & , t \ge t_1 - t_0 \end{cases}$$

This shows controllability from 0 to $t_1 - t_0$.
The other direction work analogously.

2.) Let $\mathfrak{B}(P)$ be controllable from t_0 to t_1.
Let $y_0, y_1 \in \mathfrak{B}(SPT)$, i.e., for $i = 1,2$

$$0 \; \cancel{=} \; (SPT)(\tfrac{d}{dt})\, y_i$$

$$\Rightarrow \quad 0 = P(\tfrac{d}{dt}) \underbrace{\left[T(\tfrac{d}{dt})\, y_i \right]}_{=: z_i}.$$

$\Rightarrow z_0, z_1 \in \mathfrak{B}(P)$

$\Rightarrow \exists\, z \in \mathfrak{B}(P)$ such that

$$z(t) = \begin{cases} z_0(t) & , t \le t_0 \\ z_1(t) & , t \ge t_1 \end{cases}.$$

Set $y := T^{-1}(\tfrac{d}{dt})\, z$ so that

$$y(t) = T^{-1}(\tfrac{d}{dt})\, z(t) = \begin{cases} T^{-1}(\tfrac{d}{dt})\, z_0(t) & , t \le t_0 \\ T^{-1}(\tfrac{d}{dt})\, z_1(t) & , t \ge t_1 \end{cases}$$

$$= \begin{cases} y_0(t) & , t \le t_0 \\ y_1(t) & , t \ge t_1 \end{cases},$$

which shows controllability of $\mathfrak{B}(SPT)$ from t_0 to t_1.

Task 3:

Using Lemma 1.9 one finds that $\mathcal{L}$, $\mathcal{N}$, $\mathcal{M}$ have no zeros but the blocks of type

$$\mathcal{J}_{\rho_j}(\lambda) = \begin{bmatrix} \lambda-\lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda-\lambda_j \end{bmatrix} \in \mathbb{C}[\lambda]^{\rho_j \cdot \rho_j}$$

we have $\mathcal{Z}(\mathcal{J}_{\rho_j}) = \{\lambda_j\}$.

Since the zeros of $\lambda F + G$ and the zeros of its Kronecker canonical form are the same we have that

$$\mathcal{Z}(\lambda F + G) = \bigcup_{j=1}^{u} \mathcal{Z}(\mathcal{J}_{\rho_j})$$

$$= \{\lambda_1, \dots, \lambda_u\}$$

is the union of all zeros of the blocks $\mathcal{J}_{\rho_1}, \dots, \mathcal{J}_{\rho_u}$.

Task 4:

The blocks of type $\mathcal{L}$ and $\mathcal{N}$ are left prime.

The blocks of type $\mathcal{M}$ and $\mathcal{N}$ are right prime.

The blocks of type $\mathcal{J}$ are neither.

Task 5:

"$\Leftarrow$" Let S be a polynomial right inverse and $z \in \mathfrak{B}(P)$

$\Rightarrow P(\frac{d}{dt}) z = 0 \Rightarrow \underbrace{S(\frac{d}{dt}) P(\frac{d}{dt})}_{= I} z = \underbrace{S(\frac{d}{dt}) \cdot 0}_{= 0}$

$\Rightarrow z = 0 \Rightarrow \mathfrak{B}(P) \subseteq \{0\}$

$\Rightarrow \mathfrak{B}(P) = \{0\}$

"$\Rightarrow$" ~~Pf~~ Assume to the contrary that there was a $\lambda_0 \in \mathbb{C}$ with $\operatorname{rank}(P(\lambda_0)) < q$. Then there exists a $\alpha_0 \neq 0$ such that

$$P(\lambda_0)\alpha_0 = 0.$$

Set $z(t) := e^{\lambda_0 t}\alpha_0$. Using Task 1 we find that

$$P(\tfrac{d}{dt}) z(t) = P(\lambda_0) e^{\lambda_0 t}\alpha_0 = \underbrace{P(\lambda_0)\alpha_0}_{=0} e^{\lambda_0 t} = 0$$

$\Rightarrow z \in \mathfrak{B}(P)$ although $z \neq 0$. ↯.

Task 6:

In this case the Smith form is

$P = S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T$. Set $R := [I_r, 0]\, T$.

Task 7: If $\operatorname{rank} P(\lambda_0) = q \quad \forall \lambda_0 \in \mathbb{C}$

then also for all $\mu_0 \in \mathbb{C}$ we have

$$\operatorname{rank} P^{\sim}(\mu_0) = \operatorname{rank} P^{*}(-\bar{\mu}_0)$$

$$= \operatorname{rank} P(\underbrace{-\bar{\mu}_0}_{=:\lambda_0}) = q,$$

i.e., that $P^{\sim}$ is left prime.

Task 8:

If there was a block of type $\mathcal{L}$ or $\mathcal{J}$ then $\mathcal{F}$ could not have full row rank.

Task 9: (With the notation from the handout)

The number of inputs is

$$q - \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$$

$$= q_1 + p_2 + q_3 - \overbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} \lambda F_1 + G_1}^{=p_1} - \overbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} \lambda F_2 + G_2}^{=p_2}$$

$$- \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)} \lambda F_3 + G_3}_{=q_3}$$

$$= q_1 - p_1 .$$

For the Kalman decomposition case, i.e., if one computes the Kalman decomposition and the canonical form in the handout for

$$\lambda[I, 0] - [A, B] =: \lambda F + G$$

then we have (with the corresponding notations)

$$p_3 = q_3 = 0, \qquad p_2 = n - r$$

$$p_1 = r, \qquad q_1 = r + m.$$

Task 10:

Block of type $\mathcal{L}_3$: (omitting minus signs)

Since the leading matrix has full row rank the "right prime reduction" is already done. Then proceed as

$$\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}, \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \to \left(\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}\right)$$

$$\to \left(\begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \to \left(\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}\right)$$

$$\to \left(\begin{bmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix}\right) \to \left(\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}\right)$$

Block of type $\mathcal{J}_3$:

Is alredy in canonical form

$$\mathcal{J}_3(\lambda) = \lambda \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & \\ & \lambda_j & 1 \\ & & \lambda_j \end{bmatrix} =: \lambda F_j + G_j$$

since F_g is invertible

Block of type $\mathcal{M}_3$:

$$\left(\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right) \to \left(\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right) \to \left(\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right)$$

$$\to \left(\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right) \to \left(\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right)$$

$\Rightarrow$ after three steps in the "right prime reduction" we find that the blocks of type $\mathcal{M}_3$ are already in the wanted form.

Blocks of type $\mathcal{M}$ give the same result

Task 11:

1. For all $\lambda_0 \in \mathbb{C}$ we have

$$\operatorname{rank} U(\lambda_0) = \operatorname{rank} \begin{bmatrix} U_1(\lambda_0) & \tilde{U}(\lambda_0) \\ 0 & U_2(\lambda_0) \end{bmatrix}$$

$$\geq \operatorname{rank}(U_1(\lambda_0)) + \operatorname{rank}(U_2(\lambda_0))$$

$$= p_1 + p_2$$

2. Let $[U_1, U_1'] =: V_1$ and $[U_2, U_2'] =: V_2$ be unimodular. Set

$$U' := \begin{bmatrix} U_1' & 0 \\ 0 & U_2' \end{bmatrix}. \text{ Then}$$

$$[U, U'] = \begin{bmatrix} U_1 & \tilde{U} & U_1' & 0 \\ 0 & U_2 & 0 & U_2' \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & U_1' & \tilde{U} & 0 \\ 0 & 0 & U_2 & U_2' \end{bmatrix} \underbrace{\begin{bmatrix} I & & & \\ & 0 & I & \\ & I & 0 & \\ & & & I \end{bmatrix}}_{=:P}$$

$$= \begin{bmatrix} V_1 & \tilde{V} \\ 0 & V_2 \end{bmatrix} P,$$

by setting $\tilde{V} := [\tilde{U}, 0]$. Furthermore, $[U, U']$ is unimodular if and only if $\begin{bmatrix} V_1 & \tilde{V} \\ 0 & V_2 \end{bmatrix}$ is, i.e., if ~~and~~ for example

There exists a polynomial matrix X with

$$\underline{I} = \begin{bmatrix} V_1^{-1} & X \\ 0 & V_2^{-1} \end{bmatrix} \begin{bmatrix} V_1 & \tilde{V} \\ 0 & V_2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{I} & V_1^{-1}\tilde{V} + XV_2 \\ 0 & \underline{I} \end{bmatrix}.$$

However, choosing $X := -V_1^{-1}\tilde{V}V_2^{-1}$ one can see that this is true.

3. If S_1, S_2 are the left inverses one can ~~see~~ find similar to Z. that

$$S := \begin{bmatrix} S_1 & X \\ 0 & S_2 \end{bmatrix},$$

with $X := -S_1 \tilde{U} S_2$, is a left inverse of U.

Task 12:

Since for all λ_0 we have

$$\operatorname{rank}(\lambda_0 F + G) = \operatorname{rank}\begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda_0 L & 0 & -1 \end{bmatrix} \cdot R$$

$$= \operatorname{rank}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & R & R & 1 \\ 0 & \lambda_0 L & 0 & -1 \end{bmatrix} = \operatorname{rank}\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ & R & 1 & R \\ & & -1 & \lambda_0 L \end{array}\right] = 3$$

Lemma 1.9 implies that $\mathcal{Z}(P) = \emptyset$. Thus $\mathfrak{B}(P)$ is controllable by ~~Def~~ Definition 2.14.

Task 13:

1.) The system

$$\left\{ \begin{bmatrix} q \\ f \end{bmatrix} \in \mathcal{C}_\infty^2 \;\middle|\; m\ddot{q} + d\dot{q} + kq = f \right\}$$

$$= \left\{ \begin{bmatrix} q \\ f \end{bmatrix} \in \mathcal{C}_\infty^2 \;\middle|\; \left[\left(\tfrac{d}{dt}\right)^2 m + \left(\tfrac{d}{dt}\right) d + k, -1 \right] \begin{bmatrix} q \\ f \end{bmatrix} = 0 \right\}$$

with the Definition $P(\lambda) := [\lambda^2 m + \lambda d + k, -1]$ can be written as $\mathfrak{B}(P)$.

Since for all $\lambda_0 \in \mathbb{C}$ we have

$$\operatorname{rank}[P(\lambda_0)] = \operatorname{rank}[\lambda_0^2 m + \lambda_0 d + k, -1] = 1$$

Lemma 1.9 again implies controllability of the system.

2.) Here the system is $\mathfrak{L}(P)$ with

$$P(\lambda) := \left[\begin{array}{c|c|c} \lambda^2 m_1 + \lambda d_1 + k_1 & & -1 \\ \hline & \lambda^2 m_2 + \lambda d_2 + k_2 & -1 \end{array}\right].$$

Since the set of zeros does not change under unimodular transformations we have

$$\mathfrak{Z}(P) = \mathfrak{Z}\left(\left[\begin{array}{c|c|c} \lambda^2 m_1 + \lambda d_1 + k_1 & -(\lambda^2 m_2 + \lambda d_2 + k_2) & -1 \\ \hline 0 & 0 & -1 \end{array}\right]\right)$$

$$= \mathfrak{Z}\left(\left[\begin{array}{c|c|c} \lambda^2 m_1 + \lambda d_1 + k_1 & -(\lambda^2 m_2 + \lambda d_2 + k_2) & \\ \hline & & -1 \end{array}\right]\right)$$

$$= \mathfrak{Z}\left(\left[\underbrace{\lambda^2 m_1 + \lambda d_1 + k_1}_{=:p_1}, -(\underbrace{\lambda^2 m_2 + \lambda d_2 + k_2}_{=:p_2})\right]\right).$$

$$= \mathfrak{Z}\left(\left[\lambda^2 + \lambda\frac{d_1}{m_1} + \frac{k_1}{m_1}, \lambda^2 + \lambda\frac{d_2}{m_2} + \frac{k_2}{m_2}\right]\right)$$

Notation on exercise sheet

$$= \mathfrak{Z}\left(\left[(\lambda - \lambda_1^{(1)})(\lambda - \lambda_2^{(1)}), (\lambda - \lambda_1^{(2)})(\lambda - \lambda_2^{(2)})\right]\right).$$

This set is empty if and only if

$$\{\lambda_1^{(1)}, \lambda_2^{(1)}\} \cap \{\lambda_1^{(2)}, \lambda_2^{(2)}\} = \emptyset, \text{ i.e.,}$$

if the two polynomials p_1, p_2 have no common zeros. Since we assume that $\lambda_{1,2}^{(i)}$ is a complex conjugate pair, this is the case if and only if

$$\mathrm{Re}(\lambda_1^{(1)}) \neq \mathrm{Re}(\lambda_1^{(2)}) \quad \vee \quad \mathrm{Im}(\lambda_1^{(1)}) \neq \mathrm{Im}(\lambda_1^{(2)})$$

$$\iff \frac{d_1}{2m_1} \neq \frac{d_2}{2m_2} \quad \vee \quad \cancel{\left(\frac{d_1}{2m_1}\right)^2} - \frac{k_1}{m_1} \neq \cancel{\left(\frac{d_2}{2m_2}\right)^2} - \frac{k_2}{m_2}.$$